Differential Geometry

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Contents

1 Submanifold

Definition 1.1. *ϕ is a C^r map from $U \in \mathscr{E}$ to $V \in \mathscr{F}$, U, V are open sets.*

*1. ϕ is a **diffeomorphism** if ϕ is a bijection from U to $\phi(U)$, ϕ is of class C^r and ϕ^{-1} is also of class C^r.*

*2. If $x \in U$, ϕ is an **immersion** at x, if $D_x\phi$ is injective ($\leq$).*

*3. If $x \in U$, ϕ is a **submersion** at x, if $D_x\phi$ is surjective ($\geq$).*

Theorem 1.1 (**Inversion theorem**). *If ϕ is a C^r map from $U \in \mathscr{E}$ to $V \in \mathscr{F}$, U, V are open sets. If $x \in U$ and $D_x\phi$ is a bijection, then ϕ is a diffeomorphism in a neighborhood of x.*

Definition 1.2. *Let $\mathscr{E}$ be an affine space, (U, X) is a **chart**, when U is an open set in $\mathscr{E}$, X is a map into some open set in $\mathbb{R}^n$ and X is diffeomorphism from U to $X(U)$.*

$X = (x_1, \cdots, x_n)$ the coordinate of the chart, U the domain of the chart.

Theorem 1.2 (**Immersion theorem**). *Let $\phi : \mathscr{O} \hookrightarrow \mathscr{F}$ such that ϕ is an immersion at x, there is a neighborhood $U \ni x$ and a chart (V, X) at $\phi(x)$ with $\phi(U) \subset V$, such that $X \circ \phi$ is a restriction of an affine injective map from U to $\mathbb{R}^n$.*

Proof. Set $f = X \circ \phi$ and treat f from $\mathbb{R}^p$ to $\mathbb{R}^n$, $f = (f_1, \cdots, f_n)$. Then we define $\tilde{y} = (y_1, \cdots, y_p, y_{p+1}, \cdots, y_n)$, $y = (y_1, \cdots, y_p)$ and $F(\tilde{y}) = (f_1(y), \cdots, f_p(y), f_{p+1}(y) + y_{p+1}, \cdots, f_n(y) + y_n)$.

$$\det \frac{DF}{D\tilde{y}}(x_1, \cdots, x_p, 0, \cdots, 0) = \det \frac{Df}{Dy}(x) \neq 0.$$

Then by inverse theorem, there is a neighborhood $\tilde{U}$ of $\tilde{x} = (x_1, \cdots, x_p, 0, \cdots, 0)$ such that $G : F(\tilde{U}) \to \tilde{U}$ is the inverse of F. Note that $F(y_1, \cdots, y_p, 0, \cdots, 0) = f(y_1, \cdots, y_p)$, hence $(y_1, \cdots, y_p, 0, \cdots, 0) = G \circ f(y_1, \cdots, y_p)$ for $(y_1, \cdots, y_p, 0, \cdots, 0) \in U$ which equivalent to $(y_1, \cdots, y_p) \in \tilde{U} \cap \mathbb{R}^p =: U$, which suggests f is injective. □

Corollary 1.1. *For smooth map $\phi : M \to N$, where M and N are manifolds, if ϕ is an immersion at $x \in M$, there is a neighborhood $U \ni x$ and a chart (V, X) at $\phi(x)$ with $\phi(U) \subset V$, such that the image of $(X \circ \phi)|_U$ is an open subset in $\mathbb{R}^p \subset \mathbb{R}^n$, where $p = \dim M$, $n = \dim N$.*

Theorem 1.3 (**Submersion theorem**). *Let $\phi : \mathscr{O} \hookrightarrow \mathscr{F}$ such that ϕ is an submersion at x, there is a chart (U, Y) at x such that $\phi \circ Y^{-1}$ is a restriction of an affine surjective map.*

Proof. Similarly, consider $F(y_1, \cdots, y_n) = (f_1(y), \cdots, f_p(y), y_{p+1}, \cdots, y_n)$. □

Theorem 1.4 (**Constant rank theorem**). *Let $\phi : \mathscr{O} \hookrightarrow \mathscr{F}$, $\mathscr{O}$ open set in $\mathscr{E}$, assume that $D_y\phi$ has a constant rank for y in a neighborhood of x. There is a chart on a neighborhood (U, X) of $\phi(x)$ and (V, Y) of x, such that $X \circ \phi \circ Y^{-1}$ is (a restriction of) an affine map.*

Definition 1.3. *Submanifolds of affine spaces.*

*$M \subset \mathscr{E}$ is a **submanifold** if $\forall x \in M$, there exist a chart (U, X) at x such that $X(M \cap U)$ is an open set of a vector sub-space in $\mathbb{R}^n$.*

The dimension of M is defined to be the dimension of $X(M \cap U)$

$$\dim(M) := \dim(X(M \cap U)).$$

Theorem 1.5.

1. Let M be a submanifold in $\mathscr{E}$, if $\phi : \mathscr{O} \to \mathscr{F}$ is a diffeomorphism, $\phi(M \cap \mathscr{O})$ is a submanifold.

2. Let ϕ be a submersion along $\phi^{-1}(y)$, where $\phi : \mathscr{O} \to \mathscr{F} \ni y$. Then $\phi^{-1}(y)$ is a submanifold.

$$\dim \phi^{-1}(y) = \dim \mathscr{E} - \dim \mathscr{F}.$$

3. Let ϕ be an immersion form $\mathscr{O} \subset \mathscr{E}$ to $\mathscr{F}$ at x, then there exists an open set $U \ni x$ such that $\phi(U)$ is a submanifold.

$$\dim \phi(U) = \dim \mathscr{E}.$$

Remark 1.1. *If ϕ is diffeomorphism from U to V, if (W, X) is chart with $W \subset V$, then $\left(\phi^{-1}(W), X \circ \phi\right)$ is a chart.*

Definition 1.4. ***Tangent space*** *of a submanifold.*

Let curve $c \in C^\infty :]a, b[\to \mathscr{E}$,

$$\dot{c}(t_0) := \lim_{t \to t_0} \left(\frac{c(t) - c(t_0)}{t - t_0} \right) \in E.$$

Let M be a submanifold in $\mathscr{E}$, then

$$T_x M = \{\dot{c}(0) \text{ for curves } c : [a, b] \ni 0 \text{ in } \mathscr{E} \text{ such that } \forall t, c(t) \in M, c(0) = x\}.$$

Theorem 1.6.

1. Let ϕ be a diffeomorphism, and M a submanifold

$$T_{\phi(x)} \phi(M) = D_x \phi(T_x M).$$

2. If M is an open set in an affine subspace of $\mathscr{E}$, then $T_x M$ is the underlying vector space of M.

3. If ϕ is a submersion along $\phi^{-1}(y)$,

$$T_x \phi^{-1}(y) = \ker(D_x \phi), \qquad \phi(x) = y.$$

4. If ϕ is an immersion at x,

$$T_{\phi(x)} \phi(U) = \operatorname{Im} D_x \phi.$$

2 Manifold

Definition 2.1. *All topological spaces X considered are*

1. *Hausdorff (séparé en français).*
2. *σ-compact: Countable union of compact sets.*

Definition 2.2. *Let M be a topological space, a* ***chart*** *on M is a part (U, X) where*

1. *U is an open set in M (called the domain).*
2. *$X = (x_1, \cdots, x_n)$ is a homeomorphism from U to an open set in $\mathbb{R}^n$.*

Definition 2.3. *Two charts (U, X) and (V, Y) are C^k* ***compatible*** *if $Y \circ X^{-1} : X(U \cap V) \to Y(U \cap V)$ is a C^k diffeomorphism.*

Definition 2.4. *$f : U \to \mathbb{R}$ is a C^k function w.r.t. X if $f \circ X^{-1}$ is C^k.*

Proposition 2.1. *If (U, X) and (V, Y) are C^k compatible, f defined on $U \cap V$. Then f is C^k w.r.t. $(U \cap V, X) \iff f$ being C^k w.r.t. $(U \cap V, Y)$.*

Proof.

$$f \circ X^{-1} = (f \circ Y^{-1}) \circ (Y \circ X^{-1}).$$

□

Definition 2.5. *If M is a topological space. An* ***atlas*** *on M is a collection of charts $\mathscr{U} = \{(U_i, X_i)\}_{i \in I}$ such that (i) $\bigcup U_i = M$ and (ii) all charts are pairwise C^k compatible.*

Definition 2.6. *The atlases $\mathscr{U}$ and $\mathscr{V}$ are C^k* ***compatible*** *if any chart of $\mathscr{U}$ is C^k compatible with any chart of $\mathscr{V}$. (Equivalently, $\mathscr{U} \cup \mathscr{V}$ is still an atlas.)*

Definition 2.7. *A C^k* ***manifold*** *is a topological space M (Hausdorff and σ-compact) equipped with an equivalence class (w.r.t. C^k compatibility) of atlas.*

Definition 2.8. *A C^k chart (U, X) on a manifold is a chart which is compatible with any atlas defining the manifold structure.*

Proposition 2.2. *If M is a manifold, an open set U in M is also a manifold.*

Proposition 2.3. *If M and N are manifolds, then $M \times N$ is a manifold.*

Definition 2.9. *A function on $\Sigma \subset M$ is C^k if for every $x \in \Sigma$, there exits a C^k chart on M, (U, X) with $x \in U$, $f \circ X^{-1}$ is C^k at $X(x)$.*

Example 2.1. *If (U, X) is a chart, $X = (x_1, \cdots, x_n)$. $x_1, \cdots, x_n$ are the coordinates functions on U, then x_i are smooth functions on U.*

Definition 2.10. *Let $M \xrightarrow{\phi} N$ be a map between two (smooth) manifolds. The ϕ is smooth at $x \in M$ if for every smooth function f defined on a neighborhood of $\phi(x)$, then $f \circ \phi$ is smooth at x.*

Proposition 2.4. *We have two notions of smooth map $\varphi : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$:*

(i). φ smooth as map between manifolds.

(ii). φ is smooth as a classical notion.

We will prove these two notions coincide.

Proof. $\Leftarrow$: Let f be a smooth function at $\varphi(x)$, then $f \circ \varphi$ is smooth (composition of smooth function).

$\Rightarrow$: Assume $\varphi : U \to V$ is smooth, we can find $\varphi = (\varphi_1, \cdots, \varphi_p)$, where $\varphi_i = x_i \circ \varphi$ is the coordinate functions on V. Thus φ_i is smooth, then φ is smooth.

□

Lemma 2.1. *If $f_1, \cdots, f_n$ are smooth functions $M \to \mathbb{R}$, g is a smooth function $\mathbb{R}^n \to \mathbb{R}$, then $g(f_1, \cdots, f_n)$ is a smooth function on M.*

Proposition 2.5. *ϕ is smooth at x, is equivalent to, there exits a chart (U, X) where U a neighborhood of $\phi(x)$, $X = (x_1, \cdots, x_n)$ and $x_i \circ \phi$ is smooth on $\phi^{-1}(U)$.*

Proof. $\Rightarrow$ is by definition.

$$f \circ \phi = (f \circ X^{-1}) \circ (X \circ \phi) = (f \circ X^{-1})(x_1 \circ \phi, \cdots, x_n \circ \phi).$$

□

Proposition 2.6. *$M \xrightarrow{\phi} N \xrightarrow{\psi} W$. If ϕ is smooth at x, ψ is smooth at $\phi(x)$, then $\psi \circ \phi$ is smooth at x.*

Proof. Let f be a smooth function on a neighborhood of $\psi \circ \phi(x)$, then $g = f \circ \psi$ is smooth, then $g \circ \phi$ is smooth. □

Proposition 2.7. *$\phi : M \to N$ is smooth at x, is equivalent to, there exists (U, X) a chart at x and (U, Y) at $y = \phi(x)$, such that $Y \circ \phi \circ X^{-1}$ is a smooth map.*

Proof. $\Leftarrow$: Let f be a function smooth at $\phi(x)$, we shall show that $f \circ \phi$ is smooth at x.

$$f \circ \phi \circ X^{-1} = (f \circ Y^{-1}) \circ (Y \circ \phi \circ X^{-1})$$

is smooth, hence $f \circ \varphi$ is smooth.

$\Rightarrow$: Let g be a function smooth at $\phi(x) \in Y$.

$$g \circ (Y \circ \phi \circ X^{-1}) = (g \circ Y) \circ \phi \circ X^{-1},$$

is smooth at $X(x)$. □

Exercise 2.1. *$\phi : M \to N$ is smooth at x, is equivalent to, for any (U, X) a chart at x and (U, Y) at $y = \phi(x)$, we have $Y \circ \phi \circ X^{-1}$ is a smooth map.*

Proof. Just by changing of charts. □

Exercise 2.2. *N is a submanifold of $\mathbb{R}^n$, prove that $i : N \hookrightarrow \mathbb{R}^n$ is a smooth map.*

Proof. Since N is a submanifold of $\mathbb{R}^n$, for any $x \in N$, there is a chart (U, X) such that $X(U \cap N) = \mathbb{R}^p \subset \mathbb{R}^n$. Note that $(U \cap N, X|_N)$ is a chart on N as a manifold.

Thus, $i = X^{-1} \circ X|_N$ around x is smooth. □

Exercise 2.3. *$M \times N \xrightarrow{p} M$ is a smooth map.*

Proof. For any point $(x, y) \in M \times N$, there is a chart $(U \times V, X)$ around (x, y) and a chart (V, Y) around $x \in M$, where $Y(m) = X(m, 0)$ for any $m \in M$.

Hence $Y \circ p \circ X^{-1} : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is a projection, $(x^1, \cdots, x^{m+n}) \mapsto (x^1, \cdots, x^m)$, which is smooth. Then p is smooth. □

Exercise 2.4. *If (U, X) is a chart, then X and X^{-1} are both smooth.*

Proof. For any smooth function f around $X(x) \in \mathbb{R}^n$, we need to show that $f \circ X$ is smooth around $x \in U$. We consider the chart (U, X), and we have $(f \circ X) \circ X^{-1} = f$ is smooth on $X(U)$.

For any smooth function g around $x \in U$, we need to show that $g \circ X^{-1}$ is smooth around $X(x) \in X(U)$. Since g is smooth around x, there is a chart (V, Y) around x such that $g \circ Y^{-1}$ is smooth around $Y(x)$. Now $g \circ X^{-1} = (g \circ Y^{-1}) \circ (Y \circ X^{-1})$ is a composition of smooth maps, hence smooth. □

Definition 2.11. *$\varphi : U \subset M \to V \subset N$, φ is a **diffeomorphism** if and only if φ is bijective, φ is smooth and φ^{-1} as well.*

X is a diffeomorphism if (U, X) is a chart.

Definition 2.12. *φ is a **immersion** at x, is equivalent to, there exist (U, X) and (V, Y) charts at x and $\varphi(x)$ such that $Y \circ \varphi \circ X^{-1}$ is an immersion.*

It is also equivalent to, for any (U, X) and (V, Y) charts at x and $\varphi(x)$, we have $Y \circ \varphi \circ X^{-1}$ is an immersion.

Definition 2.13. *Same definition for the **submersion**.*

Remark 2.1. *ξ is an immersion from $U \subset \mathbb{R}^n$ to $V \subset \mathbb{R}^n$, then $\phi_0 \circ \xi \circ \phi_1$ is also an immersion if ϕ_0 and ϕ_1 are diffeomorphisms.*

Example 2.2. *$M \times N \xrightarrow{p} M$ is a submersion. $M \to M \times x \subset M \times N$ is an immersion.*

Definition 2.14. *V a **submanifold** of a manifold M if $\forall x \in V$, there is a chart (U, X) at $x \in M$ such that $X(V \cap U)$ is a submanifold.*

Definition 2.15. *$\varphi : M \to N$ is an **embedding** if*

(i) φ is an injective immersion.

(ii) φ is an homeomorphism onto its image.

Example 2.3. *Topologist's sine curve.*

$\varphi : (-\infty, 0] \to \mathbb{R}^2$ *is injective and immersion but not an embedding.*

Exercise 2.5. *If M is compact and φ is an injective immersion, then φ is an embedding.*

Proof. φ brings a closed set onto a closed set in $\varphi(M)$. □

Definition 2.16. *We say φ is* ***proper*** *if $\varphi^{-1}(K)$ is compact for any compact set K.*

Example 2.4. $x \mapsto \arctan x$ *is not proper.*

Proposition 2.8. *φ is an injective immersion and φ is proper, then φ is an embedding.*

Proof. We will show that $\varphi : M \to N$ brings closed set to closed set.

First there is a collection of compact sets $\{K_n\}$ of N with $K_n \subset K_{n+1}$, such that $\cup K_n = N$. Then since φ is proper, $\varphi^{-1}(K_n)$ are compact and $M = \varphi^{-1}(N) = \varphi^{-1}(\cup K_n) = \cup \varphi^{-1}(K_n)$.

For any closed set C in M, there is a positive integer m such that $C \subset \varphi^{-1}(K_m)$, hence $\varphi(C) \subset K_m$. We show that $\overline{\varphi(C)} = \varphi(C)$.

For any $y \in \overline{\varphi(C)}$, there is a sequence $\{x_n\} \subset C$ such that $y = \lim \varphi(x_n)$. Since C is compact, there is a subsequence $\{x_{k_n}\}$ such that they converge to $x_0 \in C$. By the continuity of φ, we have

$$\varphi(x_0) = \lim \varphi(x_{k_n}) = y,$$

hence $y \in \varphi(C)$, which shows that $\overline{\varphi(C)} \subset \varphi(C)$. □

Exercise 2.6. *If φ is an embedding then $\varphi(M)$ is a submanifold, and φ is a diffeomorphism $M \to \varphi(M)$.*

Proof. By the immersion theorem, there exists $W \subset M$, $W \in V(y)$ with $\varphi(y) = x$ such that $\varphi(W)$ is a submanifold.

We can always assume by taking W smaller that $\varphi(W) \subset U$. Then we know that $\varphi(W) = \varphi(M) \cap \mathcal{O}$, where $\mathcal{O}$ open in N. Then we have $\varphi(W) = \varphi(M) \cap \mathcal{O} \cap U$. □

3 Examples

3.1 Projection Space

Definition 3.1. *Let V be a vector space with $\dim V < \infty$. A line L is a vector space in V with $\dim L = 1$. V is a vector space over any filed $\mathbb{K}$ (here $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$). We define the* ***Projective Space***

$$\mathbb{P}(V) = \{L : L \text{ lines in } V\}.$$

1. Show that $\mathbb{P}(V)$ is in bijection with $V \setminus \{0\}/\mathbb{K}^*$.

Proof. Consider $\varphi : V \setminus \{0\}/\mathbb{K}^* \to \mathbb{P}(V)$, $[v] \mapsto \mathbb{K}v$. Then it suffices to show φ is bijection, which is obvious. □

2. Define a topology on $\mathbb{P}(V)$.

Definition. U is open in $\mathbb{P}(V)$ iff $\pi^{-1}(U)$ is open, here $\pi : V \setminus \{0\} \to \mathbb{P}(V)$. □

2.1 $\mathbb{P}(V)$ is Hausdorff?

Proof. For any two different points $L, M \in \mathbb{P}(V)$, we intersect L, M with $\mathbb{S}(V)$ to get x_1, x_2, y_1, y_2. Then we can find $r > 0$ such that $B_r(x_1), B_r(x_2), B_r(y_1), B_r(y_2)$ do not intersects each other. Hence we consider the cones generated by $B_r(x_1), B_r(x_2)$ and by $B_r(y_1), B_r(y_2)$, with origin being vertex, calling C_x and C_y. Then $C_x \setminus \{0\}$ and $C_y \setminus \{0\}$ are open in $V \setminus \{0\}$ and they don't intersect. Thus, we find two separate open sets $\pi(C_x \setminus \{0\}$ and $\pi(C_y \setminus \{0\})$ in $\mathbb{P}(V)$ which contains L and M respectively.

□

2.2 $\mathbb{P}(V)$ is compact.

Proof. We know $\mathbb{S}(V)$ is compact. For any open covering $\mathscr{U} = \{U_i\}_{i \in I}$ of $\mathbb{P}(V)$, we have $\pi^{-1}(\mathscr{U}) = \{\pi^{-1}(U_i)\}_{i \in I}$ is an open covering of $V \setminus \{0\}$. Moreover, $\pi^{-1}(\mathscr{U}) \cap \mathbb{S}(V) = \{\pi^{-1}(U_i) \cap \mathbb{S}(V)\}_{i \in I}$ is an open covering of $\mathbb{S}(V)$. Since $\mathbb{S}(V)$ is compact, we have a finite subset $J \subset I$ with $\{\pi^{-1}(U_i) \cap \mathbb{S}(V)\}_{i \in J}$ being an open sub-covering of $\mathbb{S}(V)$, hence $\{U_i\}_{i \in J}$ is an open sub-covering of $\mathbb{P}(V)$.

However, this just the proof of continuous map maps a compact set to a compact set. □

3. Projective chart.

Definition. Let H be a hyperplane in V, let

$$U_H = \{L \in \mathbb{P}(V) : L \oplus H = V\}.$$

$\pi^{-1}(U_H) = V \setminus H$ is open, hence U_H is open.

$U_i = \{[x_1, \cdots, x_n] : x_i \neq 0\}$, $\phi_i : [x_1, \cdots, x_n] \mapsto \left(\frac{x_1}{x_i}, \cdots, \frac{x_n}{x_i}\right) \in \mathbb{R}^{n-1}$.

Transition map of $U_1 \to U_2$ maps $(v_1, \cdots, v_{n-1})$ to $(\frac{1}{v_1}, \frac{v_2}{v_1}, \cdots, \frac{v_{n-1}}{v_1})$.

Exercise 3.1. *Prove that $U_i \to \mathbb{K}^{n-1}$ is a homeomorphism.*

$\square$

Given a hyperplane H in V, $H = \ker \omega$, let

$$U_H = \{L : \omega|_L \neq 0\} = \{L : L \oplus H = V\}.$$

We fix $D \in U_H$, and define $\phi_H : U_H \to H$ as follows. For any $L \in U_H$, let u_L be such that $\omega(U_L) = 1$, then we define $\Phi_{H,D}(L) = u_L - u_D$.

Now we calculate $\Psi = \Phi_{H_1,D_1} \circ \Phi^{-1}_{H_0,D_0}$.

$$v \mapsto l = u_0 + v \mapsto \frac{u_0 + v}{\omega_1(u_0 + b)} - u_1.$$

We have an one-one correspondence φ:

$$\mathbb{P}(V) \to \{\text{sym projector of } \operatorname{tr} 1\}.$$

Now we want to understand $T_L\varphi$. If φ is an injective immersion, and since $\mathbb{P}(V)$ is compact, we say φ is an embedding.

We choose a hyperplane H and the unit normal vector u_0, then we consider these as a chart.

$$H \xrightarrow{\Phi^{-1}} \mathbb{P}(V) \xrightarrow{\varphi} \{\text{sym projector of } \operatorname{tr} 1\}.$$

$$w \mapsto L' \mapsto p_w.$$

The first formula is

$$\Phi^{-1}(w) = w + u_0,$$

and the second formula is, by Pythagoras theorem,

$$P_{L'}(a) = \frac{\langle a, v\rangle}{\langle v, v\rangle} v.$$

where v is a vector in L'. Then we have the formula for $\psi = \varphi \circ \Phi^{-1}$,

$$\psi(w)(a) = \frac{\langle a, u_0 + w\rangle}{\langle u_0 + w, u_0 + w\rangle}(u_0 + w).$$

Set $w = th$, for $h \in H$, we can prove that

$$D_0\psi(h)(a) = \langle a, h\rangle u_0 + \langle a, u_0\rangle h.$$

Then we have

$$D_0\psi(h)(u_0) = h \neq 0,$$

for $h \neq 0$, which shows that $D_0\psi$ is an injection, hence ψ is an immersion at 0. Hence φ is an immersion at L', with the arbitrariness of L', φ is an immersion.

3.2 Grassmannian manifold

Definition 3.2. ***Grassmannian manifold***

$$Gr_k(V) = \{P \subset V : P \text{ vector space with } \dim P = k\}.$$

3.2.1 Affine viewpoint

We fix Q as an $n-k$ dimension subspace, and set $U_Q = \{P : P \oplus Q = V\}$. We fix $P' \in W_Q$ and for any $P \in U_Q$, we can find a linear function $f : P \to Q$ such that the graph of f is P', i.e. by setting $F(x) = x + f(x)$, $x \in P$, we have $\operatorname{Im} F = P'$.

Remark 3.1. *If we want a canonical function f, we can choose P' as the perpendicular direct complement of Q.*

We can prove that F is a bijection. First F is linear, hence it suffices to show that F is injective. If there exists $x \in P$ s.t. $F(x) = 0 \in P' \subset V$, i.t., $x + f(x) = 0$. We have $f(x) = -x \in P$, hence in $P \cap Q = \{0\}$, i.e. $x = 0$.

We can show that $\{F \text{ linear} : F^{-1} : P' \to P,\ \operatorname{rank} F = k\}$ is one-to-one with U_Q.

We define $G_P = f_P \circ F_P^{-1} : P' \to Q$, choose a basis $\{e_1, \cdots, e_k\}$ of P' and we say that $(G_P(e_1), \cdots, G_P(e_k))$ gives the coordinate of the P w.r.t. F_P.

3.2.2 Matrix viewpoint

We fix a metric on V, p is projector if $p^2 = p$, then $V = \ker p \oplus \operatorname{Im} p$ and $\dim(\operatorname{Im} p) = \operatorname{tr} p$, $(\operatorname{Im} p)^{\perp} = \ker p$. We say p is symmetric if $\langle p(x), y\rangle = \langle x, p(y)\rangle$, of course a projector is symmetric.

$$Gr_k(V) \longleftrightarrow G_k = \{\text{symmetric endomorphism } (i) p^2 = p,\ (ii) \operatorname{tr} p = k.\}.$$

We first prove that this is a one-one correspondence. The only thing we need to show is that the map is an injection. For p and p' having the same ker with $p, p' \in G_k$, then there is an invertible matrix Q such that $p' = Q^{-1} p Q$ and $\ker p$ is invariant under Q.

Then for $x \in \ker p$, then $Qx \in \ker p$, hence $p'x = Q^{-1}pQx = 0$, which shows that $x \in \ker p'$. For $x \in \operatorname{Im} p$, we have $Qx \in \operatorname{Im} p$, hence $p'x = Q^{-1}pQx = Q^{-1}Qx = x$. Thus $p' = p$.

Let W be the space of symmetric endomorphism of V, then $\dim W$ is $\frac{n(n+1)}{2}$, define $G_k = \{p : p^2 = p, \operatorname{tr} p = k\}$. G_k is defined by equation $W \xrightarrow{\Phi} W \times \mathbb{R}$, $f \mapsto (f^2 - f, \operatorname{tr} f - k)$, $G_k = \Phi^{-1}(0)$.

Question: understand $\operatorname{rank} \Phi$ closed to $p_0 = I_k$.

We have the basis of W as $E_{ij} = \frac{1}{2}(\delta_{ij} + \delta_{ji})$, $1 \le i \le j \le n$. Consider the direction derivative respect to E_{ij} at $p_0 = I_k$:

$$\begin{aligned}\frac{\partial \Phi}{\partial E_{ij}} &= \left(\lim \frac{(I_k + tE_{ij})^2 - (I_k + tE_{ij}) - I_k^2 + I_k}{t}, \lim \frac{\operatorname{tr}(I_k + tE_{ij}) - \operatorname{tr} I_k}{t} \right) \\ &= \left(I_k E_{ij} + E_{ij} I_k - E_{ij}, \operatorname{tr} E_{ij} \right).\end{aligned}$$

Thus there are five cases for i, j:

(1) $1 \leq i < j \leq k$, in which situation, $\frac{\partial \Phi}{\partial E_{ij}} = (E_{ij}, 0)$.

(2) $1 \leq i = j \leq k$, in which situation, $\frac{\partial \Phi}{\partial E_{ij}} = (E_{ij}, 1) = (E_{ii}, 1)$.

(3) $1 \leq i \leq k < j \leq n$, in which situation, $\frac{\partial \Phi}{\partial E_{ij}} = (O, 0)$.

(4) $k < i < j \leq n$, in which situation, $\frac{\partial \Phi}{\partial E_{ij}} = (-E_{ij}, 0)$.

(5) $k < i = j \leq n$, in which situation, $\frac{\partial \Phi}{\partial E_{ij}} = (-E_{ij}, 1) = (-E_{ii}, 1)$. Only the third case doesn't contribute to the rank $D_{I_k}\Phi$. Thus $\mathrm{rank}(D_{I_k}\Phi) = \frac{n(n+1)}{2} - k(n-k)$, hence $\dim \mathrm{Ker}(\Phi) = k(n-k)$.

4 Partition of unity

Definition 4.1. *Let X be a topological space, f continuous on X*

$$\mathrm{Supp}(f) := \{x : f(x) \neq 0\} = \bigcap_{f=0 \ \ on \ \ X\setminus F, F \ \ closed} F.$$

Example 4.1. *There is a function $f : \mathbb{R} \to \mathbb{R}$ smooth and*

(i) $\mathrm{Supp}(f) \subset]-1, 1[$,

(ii) $f \equiv 1$ on a neighborhood of 0,

(iii) f is even,

(iv) $0 \leq f \leq 1$.

Lemma 4.1. *Let X be a manifold, K a compact set, U open set, then there exists an open set $V \subset U$ with $K \subset V \subset U$ and there is a function φ smooth on X s.t.*

(i) $\mathrm{Supp}(\varphi) \subset U$,

(ii) $\varphi \equiv 1$ on V,

(iii) $0 \leq \varphi \leq 1$.

Proof. First case $K = \{x\}$. We can find a chart $(\mathcal{O}, \varphi)$ such that $\mathcal{O} \subset U$ and $\varphi(\mathcal{O}) \supset \overline{B(0,1)}$ and $\varphi(x) = 0$. We define X_x on X by

$$X_x(y) = \begin{cases} 0, & \text{if } y \notin \mathcal{O}; \\ \xi\left(\|\varphi(y)\|^2\right), & \text{if } y \in \mathcal{O}. \end{cases}$$

We prove that X_x is smooth: $X_x|_{\mathcal{O}}$ is smooth by definition; if $y \notin \mathcal{O}$, we know that $\exists V$ around y such that $V \cap \varphi^{-1}(\overline{B(0,1)}) = \varnothing$, then $X_x = 0$ on V.

By conclusion $X_x \equiv 1$ on $V(x)$ (on a neighborhood of x) with $\mathrm{Supp}(X_x) \subset \mathcal{O} \subset U$.

Let K be a compact set, $K \subset U$. For any $x \in K$, we choose $V_x \in V(x)$ and X_x such that $\mathrm{Supp}(X_x) \subset V_x \subset U$, $X_x \equiv 1$ on $W_x \in V(x)$. Now $\{W_x\}_{x \in K}$ is an open covering of K, hence we have a finite covering $W_{x_1}, \cdots, W_{x_n}$.

Define $\psi_0 = \sum X_{x_i}$, then

$$\mathrm{Supp}(\psi_0) \subset V := \bigcup_{i=1}^{n} V_{x_i} \subset U.$$

Moreover, $\psi_0 \geq 1$ on $\bigcup\limits_{i=1}^{n} W_{x_i} \supset K$.

Now we need to cut off ψ_0, define $\psi = f \circ \psi_0$, where $f : [0, \infty[\to [0, 1]$ smooth such that

$$f(x) \begin{cases} \equiv 0, & \text{on } [0, \frac{1}{2}] \\ \equiv 1, & \text{on } [1, \infty[\end{cases}.$$

□

Definition 4.2. *Let X be a topology space, let $\{W_\alpha\}_{\alpha\in A}$ be a covering of X. A partition of unity is a collection of function $\{X_\alpha\}_{\alpha\in A}$ such that*

(i) $\mathrm{Supp}(X_\alpha) \subset W_\alpha$, $X_\alpha(X) \subset [0,1]$.

(ii) Given x in X, only finitely many α are such that $X_\alpha(x) \neq 0$.

(iii) $\sum\limits_{\alpha\in A} X_\alpha = 1$.

Definition 4.3. *A covering $\{U_i\}_{i\in I}$ is locally finite, iff $forall x \in X$, $\exists V \in V(x)$ such that $\{i : U_i \cap V \neq \varnothing\}$ is finite.*

Definition 4.4. *Let $\{U_i\}_{i\in I}$ be a covering, a covering $\{W_j\}_{j\in J}$ is a subcovering if for any $j \in J$, there is $i \in I$ such that $W_j \subset U_i$.*

Proposition 4.1. *Let X be a topological space such that X is locally compact and σ-compact, then for any $\{U_i\}$ covering, there is a locally finite subcovering.*

Theorem 4.1 (Partition of unity)**.** *Let X be a manifold and $\{W_\alpha\}_{\alpha\in A}$ be a locally finite covering, then there is a partition of unity for W_α.*

Theorem 4.2 (Whitney)**.** *Let M be a manifold (compact), then there exists N and an embedding of M into $\mathbb{R}^N$.*

Proof. Let $(U_i, \varphi_i)_{i=1,\cdots,p}$ be a finite atlas for M. Assume $\dim M = n$, here we will set $N = pn + p$.

We extend $\varphi_i : U_i \to \mathbb{R}^n$, 0 outside U_i. Now this is not continuous.

Let $V_i \subset \overline{V_i} \subset U_i$ be open sets with $\cup V_i = M$. For example, set $K_i = M \setminus \bigcup\limits_{j\neq i} U_i$ and V_i a neighborhood of K_i.

Let ξ_i be a smooth function with $\mathrm{Supp}\,\xi_i \subset U_i$ and $\xi_i \equiv 1$ on V_i. Define

$$\Phi = (\xi_1\varphi_1, \cdots, \xi_p\varphi_p, \xi_1, \cdots, \xi_p)$$

a smooth function.

Let prove Φ is injective, assume that $\Phi(x) = \Phi(y)$. There exists i_0 such that $\xi_{i_0}(x) \neq 0$ (because $x \in V_{i_0}$, hence $\xi_{i_0}(y) \neq 0$, hence $y \in U_{i_0}$. Therefore

$$\xi_{i_0}(x)\varphi_{i_0}(x) = \xi_{i_0}(y)\varphi_{i_0}(y),$$

hence $\varphi_{i_0}(x) = \varphi_{i_0}(y)$, hence $x = y$.

Let's prove that Φ is an immersion. Let $x \in X$, there is i_0, $x \in V_{i_0}$, then $\Phi|_{V_{i_0}} = (\cdots, \varphi_{i_0}, \cdots)$ is an immersion. □

Remark 4.1.

Whitney: every compact manifold of dimension n can be embedded in $\mathbb{R}^{2n+1}$, immersed in $\mathbb{R}^{2n}$.

Source: Milnor, Topology from the differential viewpoint.

Cohen: immersed in $\mathbb{R}^{2n-a(n)}$, where $a(n) = \#\{1$ in the binary system decomposition of $n\}$: $4 = 100...$

5 Cotangent space

Definition 5.1. ***Differential*** *of a function.*

Let $f : M \to \mathbb{R}$ smooth function at x. We say "$d_x f = 0$" if the following equivalent statement are true

(i) $\exists (U, X)$ at x such that $d_{X(x)}(f \circ X^{-1}) = 0$.

(ii) $\forall (V, Y)$ at x, $d_{Y(x)}(f \circ Y^{-1}) = 0$.

Proof.

$$f \circ Y^{-1} = (f \circ X^{-1}) \circ (X \circ Y^{-1}).$$

$$d_{Y(x)} f \circ Y^{-1} = \left(d_{X(x)}(f \circ X^{-1})\right) \circ D_{Y(x)}\psi.$$

□

Proposition 5.1. *If $f = g$ on a neighborhood of x, then $d_x(f - g) = 0$.*

Exercise 5.1. *$f : U \subset \mathbb{R}^n \to \mathbb{R}$, $d_x f = 0 \iff f = f(x) + \sum_{i=1}^{k} \varepsilon_i \cdot f_i^2$, where $f_i(x) = 0, \varepsilon_i = \pm 1$.*

Proof. This reminds me the famous Morse Lemma: If x is a non-degenerate critical point for f, then there is a local coordinate system $(y^1, \cdots, y^n)$ in a neighborhood U of x with $y^i(x) = 0$ such that

$$f = f(x) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2,$$

where λ is the index of f at x.

However, here x is a critical point of f but not necessarily non-degenerate. Thus we may need to make some minor modification of the proof of Morse lemma.

Without loss of generality, we set $x = 0$ and $f(x) = 0$. Here we introduce a lemma:

Lemma 5.1. *Let f be a smooth function in a convex neighborhood V of 0 in $\mathbb{R}^n$, with $f(0) = 0$. Then*

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_n),$$

for some suitable smooth function g_i defined in V, with $g_i(0) = \frac{\partial f(0)}{\partial x_i}$.

Proof.

$$f(x_1, \cdots, x_n) = \int_0^1 \frac{df(tx_1, \cdots, tx_n)}{dt} dt = \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i} dt.$$

With this lemma, we could find g_i with $g_i(0) = \frac{\partial f}{\partial x_i} = 0$. Applying again this lemma to the g_i, then we have h_{ij} s.t.

$$g_i(x_1, \cdots, x_n) = \sum_{j=1}^{n} x_j h_{ij}(x_1, \cdots, x_n).$$

Hence it follows that

$$f(x_1,\cdots,x_n)=\sum_{i,j}x_ix_jh_{ij}(x_1,\cdots,x_n)=\sum_{i,j}\left(\frac{x_i+x_jh_{ij}}{2}\right)^2-\left(\frac{x_i-x_jh_{ij}}{2}\right)^2.$$

□

Exercise 5.2. *$f:U\subset\mathbb{R}^n\to\mathbb{R}$, $d_xf=0 \iff f=f(x)+\sum_{i=1}^{k}\varepsilon_i\cdot f_i\cdot g_i$, where $f_i(x)=0$, $g_i(x)=0,\varepsilon_i=\pm1$.*

Proof. aaa □

Let U be a neighborhood of x, let

$$\mathscr{E}(U)=\{f:\ \text{smooth on } U\}.$$

$$\mathscr{F}(U)=\{f:\ \text{smooth on } U \text{ and } \text{“}d_xf=0\text{”}\}.$$

Let's consider the vector space $\mathscr{E}(U)/\mathscr{F}(U)$.

Proposition 5.2. *$\mathscr{E}(U)/\mathscr{F}(U)$ does not depend on U.*

Let $V\subset U$ and $\mathscr{E}(U)\to\mathscr{E}(V)$ is the restriction. We claim that

Proposition 5.3. *$\Phi:\mathscr{E}(U)/\mathscr{F}(U)\to\mathscr{E}(V)/\mathscr{F}(V)$ is an isomorphism as a vector space.*

Proof.

(i) This is a linear map.

(ii) Φ is injective.

(iii) Let f be a function on V, let h be a smooth function defined on M with $h\equiv1$ on a neighborhood of x and $\operatorname{Supp}h\subset V$. We define $\tilde f=hf$ on V, $\tilde f=0$ outside V. Then $\tilde f=f$ on a neighborhood of x, then “$d(\tilde f-f)=0$”, $\tilde f=f$ in $\mathscr{E}(V)/\mathscr{F}(V)$. Now $\tilde f$ is the restriction of a function defined on U, thus $\Phi(\tilde f)=f$, which shows that Φ is surjective. □

Definition 5.2. *Cotangent space $T_x^*M=\mathscr{E}(U)/\mathscr{F}(U)$.*

Definition 5.3. *Given f defined on $U\in V(x)$, $d_xf\in T_x^*M$ is the projection of f in $\mathscr{E}(U)/\mathscr{F}(U)$.*

Remark 5.1. *Check that “$d_xf=0$” $\Leftrightarrow d_xf=0$.*

Proposition 5.4.

(i) $d_x:f\to d_xf$ is a linear map.

(ii) $d_x(fg)=f(x)d_xg+d_xfg(x)$.

Proof. Set $h=fg-f(x)g-g(x)f$, we just want to prove that $d_xh=0$.

Let us find a chart (U,X), $\tilde f=f\circ X^{-1}$, $\tilde g=g\circ X^{-1}$ and $\tilde h=h\circ X^{-1}$. Let $x_0=X(x)$, then

$$\tilde h=\tilde f\tilde g-\tilde f(x_0)\tilde g-\tilde g(x_0)\tilde f,$$

then by the Leibnitz rule, $d_{x_0}\tilde h=0$. □

Proposition 5.5. *If (U, X) is a chart at x, $X = (x_1, \cdots, x_n)$, then $(d_x x_1, \cdots, d_x x_n)$ is a basis of $T_x^* M$.*

Moreover, if $f = F(x_1, \cdots, x_n)$, then

$$d_x f = \sum_i \frac{\partial F}{\partial x_i} d_x x_i.$$

Proof. Setting $y_0 = X(x)$, $\lambda_i = \frac{\partial F}{\partial x_i}$, let us consider

$$f - \sum_i \lambda_i x_i = h,$$

Claim 1, $d_x h = 0$.

Let $\tilde{h} = h \circ X^{-1}$, $\tilde{f} = f \circ X^{-1} = F$ and $\tilde{x_i} = x_i \circ X$. Then

$$\tilde{h} = F - \frac{\partial F}{\partial x_i} \tilde{x_i},$$

by differential calculus $d_{x_0} \tilde{h} = 0$.

Then $d_x f = \sum_i \lambda_i d_x x_i$, hence $f \in \mathrm{Span}(d_x x_1, \cdots d_x x_n)$

Claim 2, $d_x x_k$ are independent.

Assume that $\sum_i \lambda_i d_x x_i = 0$, iff $d_x(\sum_i \lambda_i d_x x_i) = 0$, iff $d_x(\sum_i \lambda_i d_x \tilde{x_i} = 0$, iff $\lambda_i = 0$ for any i. ☐

Definition 5.4. ***Partial derivatives.***

If f is smooth around x, (U, X) is a chart,

$$d_x f = \sum_i^n \frac{\partial f}{\partial x_i} d_x x_i,$$

we just says $\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial x_i}$.

Exercise 5.3. *$d_x \lambda = 0$ if λ is constant on $V(x)$.*

Definition 5.5. ***Tangent space.***

$T_x M$ is the dual of $T_x^* M$, elements of $T_x M$ are called tangent vectors.

If $(X, x_1, \cdots, x_n)$, we have a basis $\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$ of $T_x M$ given as the dual basis of $(d_x x_1, \cdots, d_x x_n)$.

Remark 5.2. *Note that $\frac{\partial}{\partial x_i}$ does not only depend on x_i, but also on the whole choice of basis.*

Example 5.1. *Let c be smooth curve at x in M,*

$$c :]-1, 1[\to M, \quad c(0) = x.$$

the tangent vector to c at x is defined by

$$\langle \omega \mid \dot{c}(0) \rangle = \omega\left(\dot{c}(0)\right) = \frac{d}{dt}\Big|_{t=0} (f \circ c),$$

for any ω in $T_x^* M$ and for any f such that $d_x f = \omega$.

Exercise 5.4. *If $d_x f = d_x g$, then $\frac{d}{dt}\big|_{t=0}(f \circ c) = \frac{d}{dt}\big|_{t=0}(g \circ c)$.*

Proposition 5.6. *Let $X = (x_1, \cdots, x_n)$ be coordinates at x, let $X \circ c = (c_1, \cdots, c_n)$, then*

$$\dot{c}(0) = \sum_i^n \dot{c}_i(0) \frac{\partial}{\partial x_i}.$$

Definition 5.6. *Let γ_0, γ_1 be two curves through x, we define γ_0 and γ_1 are tangent at x iff $\dot{\gamma_0}(0) = \dot{\gamma_1}(0)$.*

Proposition 5.7. *Let $\phi : M \to N$ be smooth, if γ_0 and γ_1 are tangent at x, then $\phi \circ \gamma_0$ and $\phi \circ \gamma_1$ are tangent at $\phi(x)$.*

Proof. Let X be coordinates around x,

$$\dot{\gamma_0}(0) = \dot{\gamma_1}(0) \iff (X \dot{\circ} \gamma_0)(0) = (X \dot{\circ} \gamma_1)(0).$$

Let Y be coordinates around $y = \phi(x)$, we need to show

$$(\phi \dot{\circ} \gamma_0)(0) = (\phi \dot{\circ} \gamma_1)(0),$$

which is equivalent to

$$(Y \circ \dot{\phi} \circ \gamma_0)(0) = (Y \circ \dot{\phi} \circ \gamma_0)(0),$$

$$\iff (Y \circ \phi \circ X^{\dot{-}1} \circ X \circ \gamma_0)(0) = (Y \circ \phi \circ X^{\dot{-}1} \circ X \circ \gamma_0)(0).$$

We write ψ for $Y \circ \phi \circ X^{-1}$ and $c_i = X \circ \gamma_i$.

There $(\psi \dot{\circ} c_i)(0) = D_{X(x)}\psi\left(\dot{c_i}(0)\right)$, but since $\dot{\gamma_0}(0) = \dot{\gamma_1}(0)$, then $\dot{c_0}(0) = \dot{c_1}(0$, then

$$D_{X(x)}\psi\left(\dot{c_0}(0)\right) = D_{X(x)}\psi\left(\dot{c_1}(0)\right).$$

□

Definition 5.7. *Let ϕ be smooth from M to N. Then $T_x\phi$ is the unique linear map $T_xM \to T_{\phi(x)}N$ such that $T_x\phi\left(\dot{c}(0)\right) = (\phi \dot{\circ} c)(0)$.*

Sometimes $T_x\phi$ is written as $D_x\phi$.

Proposition 5.8.

(i) $d_x(f \circ \phi) = d_{\phi(x)}f \circ T_x\phi$.

(ii) $T_x(\phi \circ \psi) = T_{\phi(x)}\phi \circ T_x\psi$.

(iii) If $(x_1, \cdots, x_p)$ are coordinates at x, $(y_1, \cdots, y_n)$ are coordinates at $\phi(x)$, then the matrix of $T_x\phi$, in $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}$ is of a coefficients $\frac{\partial \phi_j}{\partial x_i}$, where $Y \circ \phi = (\phi_1, \cdots, \phi_n)$.

Theorem 5.1.

(i) ϕ is a diffeomorphism on $V(x)$ iff $T_x\phi$ is invertible (just by local immersion theorem).

(ii) ϕ is an immersion iff $T_x\phi$ is injective.

(iii)ϕ is an submersion iff $T_x\phi$ is surjective.

6 Differential forms

6.1 1-form.

Let $T^*M = \bigsqcup_{x \in M} T_m^*M$.

Definition 6.1. *A* ***differential form*** *of degree* 1 *is* $\omega : M \to T^*M$ *such that* $\omega(m) \in T_m^*M$.

Example 6.1.

(i) If f *is function,* $df(m) = d_m f$ *is a differential form. Such a form is called* ***exact****.*

(ii) The space of differential 1*-forms is a vector space.*

(iii) If α *is a* 1*-form,* f *a function on* M*, then* $f \circ \alpha : m \mapsto f(m) \cdot \alpha_m$ *is a* 1*-form.*

(iv) If $(x_1, \cdots, x_n)$ *are coordinates in* M *on* $V(x)$*, then*

$$\omega = \sum_{i=1}^{n} \omega_i dx_i,$$

on a neighborhood of x*, where* ω_i *are functions.*

Definition 6.2. ω *is a* ***smooth*** 1*-form if in every* x *in* M *we can find* $(x_1, \cdots, x_n)$ *on a neighborhood of* x*, such that* $\omega = \sum_{i=1}^{n} \omega_i dx_i$ *with* ω_i *smooth functions.*

Remark 6.1.

(i)If f *is smooth, then*

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i,$$

with $\frac{\partial f}{\partial x_i}$ *are smooth, hence* df *is a smooth* 1*-form.*

(ii) If $\omega = \sum_i \omega_i dx_i$ *with* ω_i *smooth, then for any coordinates* $(y_1, \cdots, y_n)$*, we have*

$$\omega = \sum_i \eta_i dy_i,$$

with η_i *smooth. Since* $\eta_i = \sum_j \omega_j \frac{\partial x_j}{\partial y_i}$*.*

Proposition 6.1. *The space* $\Omega^1(M)$ *of smooth differential forms on* M *is a vector space, and it is also a modules over* $C^\infty(M)$*.*

Integration of 1-forms

Let $\omega \in \Omega^1(M)$, let c be a curve $[a, b] \to M$, we define

$$\int_c \omega := \int_a^b \omega\left(\dot{c}(t)\right) dt.$$

If ϕ is an increasing diffeomorphism $[a, b] \to [a, b]$, then

$$\int_{c \circ \phi} \omega = \int_c \omega.$$

If ϕ is decreasing, $\int_{c\circ\phi}\omega = -\int_c \omega$.

When is a form exact? (at least locally).

$\omega = \sum \omega_i dx_i$ and $\omega = df$, then $\frac{\partial \omega_i}{\partial x_j} = \frac{\partial \omega_j}{\partial x_i}$.

We will introduce $\Omega^2(M)$ forms of degree 2.

$$d : \Omega^1(M) \to \Omega^2(M),$$

such that $d \circ d(f) = 0$.

6.2 Review of linear algebra

Let E be a vector space of finite dimension m, an **exterior form** of degree p is α, such that $\alpha : E^p \to \mathbb{K}$,

(i) α is multilinear.

(ii) $\alpha(u_{\sigma(1)}, \cdots, u_{\sigma(p)}) = (-1)^{\varepsilon(\sigma)}\alpha(u_1, \cdots, u_p), \sigma \in \mathfrak{S}_p$.

Antisymmetric 2-forms, $\alpha(u, v) = -\alpha(v, u)$.

It is enough to check (ii) when σ is a transposition.

Facts: we denote by $\bigwedge^p(E^*) = \{\text{the space of exterior } p \text{ forms}\}$, $\bigwedge^p(E^*)$ is a vector space. $\bigwedge^*(E^*) = \bigoplus_{p=0}^{\infty} \bigwedge^p(E^*)$. By convention $\bigwedge^0(E^*) = \mathbb{R}$.

Remark 6.2. *Map $E \to F$ is a subset of $E \times F$, $\varnothing \to F$, subset of $\varnothing \times F$. Note that the empty set have the subset, itself!*

If $(e_1, \cdots, e_m)$ is a basis of E, and $(e^1, \cdots, e^m)$ the dual basis of E^*. If $I = (i_1, \cdot, i_p)$ with $i_1 < \cdots < i_p$ then ω_I defined by

$$\begin{cases} \omega_I(e_{i_1}, \cdots, e_{i_p}) = 1, \\ \omega_I(e_{j_1}, \cdots, e_{j_p}) = 0, \text{otherwise}. \end{cases} .$$

defines a basis of $\bigwedge^p(E^*)$.

$\dim \bigwedge^p(E^*) = 0$ if $p > m$, $\dim \bigwedge^p(E^*) = \dim \bigwedge^{m-p}(E^*)$.

Exterior product

Facts: there is bilinear form $\bigwedge^p(E^*) \times \bigwedge^q(E^*) \to \bigwedge^{p+q}(E^*)$, $\alpha, \beta \to \alpha \wedge \beta$, which enjoys the following properties

(i) associativity, $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$.

(ii) $\alpha \wedge \beta = (-1)^{\deg \alpha \cdot \deg \beta} \beta \wedge \alpha$.

(iii) normalisation $\omega_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$.

Formula:

$$\alpha \wedge \beta(u_1, \cdots, u_{p+q}) = \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{\varepsilon(\sigma)} \alpha(u_{\sigma(1)}, \cdots, u_{\sigma(p)}) \beta(u_{\sigma(p+1)}, \cdots, \sigma(p+q)),$$

for $p = q = 1$, $(\alpha \wedge \beta)(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$.

Interior product

Bilinear form $E \times \bigwedge^p(E^*) \to \bigwedge^{p-1}(E^*)$, $u, \omega \mapsto i_u\omega$,

$$i_u\omega(v_1, \cdots, v_{p-1}) = \omega(u, v_1, \cdots, v_{p-1}).$$

Exercise 6.1.

$$i_u(\alpha \wedge \beta) = i_u\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge i_u\beta.$$

Proof. We can prove this just for orthonormal basis. □

Induction

$A : E \to F$ linear, $A^* : \bigwedge^p(F^*) \to \bigwedge^p(E^*)$,

$$(A^*\omega)(u_1, \cdots, u_p) := \omega(A_{u_1}, \cdots, A_{u_p}).$$

Proposition 6.2.

(i) $A^*(\alpha \wedge \beta) = (A^*\alpha) \wedge (A^*\beta)$.

(ii) $A^* \circ B^* = (B \circ A)^*$.

(iii) $A^*(i_{A(u)}\alpha) = i_u(A^*\alpha)$.

6.3 Differential forms on manifolds

Motivation: if (X, U) is a chart, ω is a 1-form on U, $\omega = \sum\limits_{i=1}^{m} \omega_i dx_i$.

$$d^X\omega := \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} \right) dx_i \wedge dx_j,$$

a 2-form on U. If $\omega = df$, then $d^X\omega = 0$.

If (X, U), (Y, V) are two charts, then

$$d^X\omega = d^Y\omega, \text{ on } U \cap V,$$

so we can well-define $d\omega$.

Converse is almost true: if $d\omega = 0$, then there is f such that $\omega = df$ (depends on the shape of manifold).

$\bigwedge^p(M) := \bigsqcup\limits_m \bigwedge^p(T_m^*M)$, a differential form $\omega : M \to \bigwedge^p(M)$ such that for all $m \in M$, $\omega_m \in \bigwedge^p(T_m^*M)$.

How to define smooth differential forms?

Observe that if (U, X) is a chart

$$d_m x_I := d_m x_{i_1} \wedge \cdots d_m x_{i_p}, \text{ where } I = (i_1, \cdots, i_p) \text{ with } i_1 < \cdots < i_p$$

is a basis of $\bigwedge^p(T_m^*M)$. Every form satisfies

$$\omega = \sum_I \omega_I^X dx_I,$$

on the chart.

Definition 6.3. *ω is smooth on M iff for every $x \in M$, there is a chart (U, X) at x such that $\forall I$, ω_I^X is smooth.*

Exercise 6.2. *If ω is smooth, then for every chart (V, X), ω_I^X is smooth.*

Then we define $\alpha \wedge \beta$ by

$$(\alpha \wedge \beta)_m := \alpha_m \wedge \beta_m, \quad m \in M.$$

$(\alpha + \beta)_m = \alpha_m + \beta_m$. $k \wedge \alpha := k\alpha$, for $k \in \mathbb{R}$ and $\alpha \in \bigwedge^p(E^*)$.

Proposition 6.3. *If α and β is smooth, then $\alpha \wedge \beta$ is smooth.*

Convention: $\bigwedge^0(M) = \bigsqcup\limits_m \left(\bigwedge^0(T_m M^*)\right) = \bigsqcup\limits_m (\mathbb{R})$, hence a 0-form is a function.

Notation:

$$\Omega^p(M) = \{\text{vector space of } C^\infty p\text{-form on } M\}.$$

hence $\Omega^0(M) = C^\infty(M)$, and the wedge product

$$\Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M).$$

Exterior differential

Definition 6.4. *A linear map $\Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M)$, $\forall p$, is an exterior differential if*

(i) if $\alpha = 0$ on $V(x)$, then $d\alpha = 0$ on $V(x)$.

(ii) $d(f d\alpha) = df \wedge d\alpha$.

(iii) df is the usual differential of a function.

Theorem 6.1. *On every manifold, there exist a unique exterior differential.*

Proof.

Uniqueness part:

Proposition 6.4. *If d is an exterior differential, then*

$$d(f dg_1 \wedge \cdots \wedge dg_p) = df \wedge dg_1 \wedge \cdots \wedge dg_p,$$

where $f, g_1, \cdots, g_p$ are functions.

Let us prove the proposition by induction on p. For $p = 1$, it is just definition (ii). Assume this is true for $p - 1$, then

$$dg_1 \wedge \cdots \wedge dg_p = d(g_1 dg_2 \wedge \cdots \wedge dg_p).$$

Then

$$d(f dg_1 \wedge \cdots \wedge dg_p) = d\left(f d(g_1 dg_2 \wedge \cdots \wedge dg_p)\right) = df \wedge dg_1 \wedge \cdots \wedge dg_p.$$

Proof is complete.

Corollary 6.1. *Uniqueness of an exterior differential.*

If $\alpha = \sum\limits_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, then $d\alpha = \sum\limits_I df_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, which indicates $d\alpha$ is uniquely determined.

Existence part:

Let (U, X) be a chart. Let us define d^X on $\Omega^*(U) = \bigoplus\limits_{p=0}^{\dim M} \Omega^p(U)$ by

$$d^X(\sum_I \omega_I dx_I) := \sum_I d\omega_I \wedge dx_I.$$

We now prove that d^X is an exterior differential. (i) and (iii) are obvious. For (ii), we need to prove $d^X(fd\omega) = df \wedge d^X\omega$.

Let $\omega = \sum\limits_I \omega_I dx_I$,

$$\begin{aligned}
d^X(fd^X\omega) &= \sum_I d^X\left(fd\omega_I \wedge dx_I\right) \\
&= \sum_I \sum_j d^X \left(f \frac{\partial \omega_I}{\partial x_j} dx_j \wedge dx_I \right) \\
&= \sum_I \sum_j d(f \frac{\partial \omega_I}{\partial x_j}) \wedge dx_j \wedge dx_I \\
&= \sum_I \sum_j \frac{\partial \omega_I}{\partial x_j} df \wedge dx_j \wedge dx_I + \sum_I \sum_j \sum_i f \frac{\partial^2 \omega_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_I \\
&= \sum_I df \wedge d\omega_I \wedge dx_I + 0 \\
&= df \wedge d^X\omega.
\end{aligned}$$

Assume that (U, X) and (V, Y) are charts then if f is defined on $U \cap V$, then $d^X(f|_{U\cap V}) = d^Y(f|_{U\cap V})$.

The existence follows, let ω be a p-form, we define $d\omega$ in the following way. Let (U, X) be a chart around $V(x)$,

$$d\omega|_U := d^X(\omega|_U),$$

this way we have defined coherently $d\omega$. □

Proposition 6.5. $d^2\alpha = 0$.

Proof. Let $f \equiv 1$, then

$$d^2\alpha = d(fd\alpha) = df \wedge d\alpha = 0.$$

□

Assume smooth map $F : M \to N$, in particular, $T_mF : T_mM \to T_{F(m)}N$, we define $F^* : \Omega(N) \to \Omega(M)$:

$$F^*\omega(u_1, \cdots, u_p) := \omega\left(TF(u_1), \cdots, TF(u_p)\right),$$

where $(F^*\omega)_m = (T_mF)^*\omega_{F(m)}$ and $u_1, \cdots, u_p \in T_mM$.

Remark 6.3. *$(T_mF)^*$ is a pull-back between two tangent spaces. The * is different from the * on F^*.*

Proposition 6.6.

(i) $d(d\alpha) = 0$,

(ii) $F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$,

(iii) $d(F^*\alpha) = F^*d\alpha$,

(iv) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge d\beta$.

Proof. We prove (iii). Note that this formula is linear in α, it is enough to prove it for $\alpha = f_1 df_2 \wedge \cdots \wedge df_k$, where $f_1, \cdots, f_k$ are functions on N. Indeed, locally every α is a linear combination of forms of this type.

For f a function,

$$\left(F^*(df)\right)_m(u) = df_{F(m)}\left(T_mF(u)\right) = \left(d_{F(m)}f \circ T_mF\right)(u) = \left(d(f \circ F)\right)_m(u),$$

which indicates $F^*df = d(F^*f)$.

Generally, for $\alpha = f_1 df_2 \wedge \cdots \wedge df_k$,

$$\begin{aligned} F^*\alpha &= F^*\left(f_1 df_2 \wedge \cdots \wedge df_k\right) \\ &= (F^*f_1)(F^*df_2) \wedge \cdots \wedge (F^*df_k). \end{aligned}$$

The second equality is induced from (ii). Hence

$$dF^*\alpha = d(F^*f_1) \wedge (F^*df_2) \wedge \cdots \wedge (F^*df_k) = F^*(d\alpha).$$

Now let us prove (iv). Again by linearity it is enough to prove it for $\alpha = f_1 df_2 \wedge \cdots \wedge df_k$, $\beta = g_1 dg_2 \wedge \cdots \wedge dg_m$.

$$\alpha \wedge \beta = f_1 g_1 df_2 \wedge \cdots \wedge f_k \wedge dg_2 \wedge \cdots \wedge dg_m.$$

$$\begin{aligned} d(\alpha \wedge \beta) &= d(f_1 g_1) df_2 \wedge \cdots \wedge dg_m \\ &= \left(g_1 df_1 + f_1 dg_1\right) df_2 \wedge \cdots \wedge dg_m \\ &= g_1 df_1 \wedge \cdots \wedge df_k \wedge dg_2 \wedge \cdots \wedge dg_m + f_1 dg_1 \wedge df_2 \wedge \cdots \wedge df_k \wedge dg_2 \wedge \cdots \wedge dg_m \\ &= g_1(df_1 \wedge \cdots \wedge df_k) \wedge dg_2 \wedge \cdots \wedge dg_m + (-1)^{k-1} f_1 df_2 \wedge \cdots \wedge df_k \wedge (dg_1 \wedge dg_2 \wedge \cdots \wedge dg_m) \\ &= d\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge d\beta. \end{aligned}$$

□

7 De Rham Cohomology

Given M a manifold, $TM = \bigsqcup_m T_m M$.

7.1 Poincaré lemma

A smooth vector filed is, locally in a chart, $\xi = \sum_{i=1}^{N} \xi_i \frac{\partial}{\partial x_i}$, where $\{\frac{\partial}{\partial x_i}\}$ is the dual of $\{dx_i\}$. We denote by $\chi^\infty(M)$ the smooth vector fields on M.

An interior product or a vector field with a form

$$\chi^\infty(M) \times \Omega^k(M) \to \Omega^{k-1}(M)$$
$$(\xi, \omega) \mapsto i_\xi \omega,$$

where $(i_\xi \omega)_m := i_{\xi_m} \omega_m$.

Example 7.1. *Let $N = M \times \mathbb{R}$. On $M \times \mathbb{R}$ there is a natural vector ∂_t,*

$$\partial_t(m, s) = \frac{d}{dt}\Big|_{t=s} c(t),$$

where $c(t) = (m, t)$.

What is the $i_{\partial_t}\alpha$? α is a form on N, locally we can find a chart (U, X) on M, and hence $(U \times \mathbb{R}, (X, t))$ on N, where $t : (m, s) \mapsto s$. In $U \times \mathbb{R}$,

$$\alpha = \sum_I f_I dx_I + dt \wedge \left(\sum_J g_J dx_J \right).$$

In general we can write $\alpha = \alpha_0 + dt \wedge \alpha_1$. We will prove $i_{\partial_t}\alpha_0 = 0$ and $i_{\partial_t}\alpha = \alpha_1$.

$$dt(\partial_t) = \frac{d}{dt}\Big|_{t=s} t(m, s) = \frac{d}{dt}\Big|_{t=s} s = 1.$$

$$dx_i(\partial_t) = \frac{d}{dt}\Big|_{t=s} x_i(m, s) = \frac{d}{dt}\Big|_{t=s} x_i(m) = 0.$$

$$\begin{aligned}
i_{\partial_t}\alpha(u_2, \cdots, u_k) &= \alpha(\partial_t, u_2, \cdots, u_k) \\
&= \alpha_0(\partial_t, u_2, \cdots, u_k) + (dt \wedge \alpha_1)(\partial_t, u_2, \cdots, u_k) \\
&= 0 + i_{\partial_t}(dt \wedge \alpha_1) \\
&= (i_{\partial_t} dt) \wedge \alpha_1(u_2, \cdots, u_k) + (-1) dt \wedge i_{\partial_t}\alpha_1(u_2, \cdots, u_k) \\
&= \alpha_1(u_2, \cdots, u_k).
\end{aligned}$$

Remark 7.1. $(e_{i_1} \wedge \cdots \wedge e_{i_p})(e^{j_1}, \cdots, e^{j_p}) = \det\{e_{i_s} e^{j_t}\}_{s,t}$.

Hence if I and J are of increasing order, it is 1 only if $I = J$, otherwise it is 0.

Definition 7.1. *We say α is* ***closed*** *if $d\alpha = 0$ and α is* ***exact*** *if there exists β such that $\alpha = d\beta$.*

We say α and β are ***cohomologuous*** *if $\alpha - \beta$ is exact, for α and β closed.*

Definition 7.2. *Let $F_0 : M \to N$ and $F_1 : M \to N$ be two smooth maps. We say F_0 is **homotopic to** F_1 if there exists $F : M \times [0,1] \to N$ such that F is smooth, $F(m,0) = F_0(m)$ and $F(m,1) = F_1(m)$.*

Remark 7.2. *Working definition $G : M \times [0,1] \to N$ is smooth if there exists a smooth map $G_0 : M \times \mathbb{R} \to N$ such that $G_0|_{M\times[0,1]} = G$.*

Definition 7.3. *We say a manifold M is **contractible**, if the identity $M \to M$ is homotopic to a constant map $K_{m_o} : M \to \{m_0\} \subset M$.*

Example 7.2. *An open ball B in $\mathbb{R}^n$, $F(x,t) = tx$ for $x \in B, t \in [0,1]$.*

Proposition 7.1. *Every compact manifold is not contractible.*

Proof. We will prove it later, or not.

My thought: For orientable compact manifold, the top Betti number is 1. For non-orientable case, we can choose the orientable double cover. □

Theorem 7.1 (Poincaré Lemma). *If M is contractible, then every closed form is exact.*

Theorem 7.2 (Homotopy Lemma). *If $\alpha \in \Omega^k(N)$ is closed on N. If F_0 and F_1 are homotopic maps $M \to N$, then $F_0^*\alpha$ and $F_1^*\alpha$ are cohomologuous.*

Proof. We use the Homotopy Lemma to prove the Poincaré Lemma.

Set $F_0 = \mathrm{id}$ and F_1 is a constant map. Since M is contractible, F_0 is homotopic to F_1. $F_0^*\alpha = \alpha$ and $F_1^*\alpha = 0$. Hence α is cohomologuous to 0, which means α is exact.

Now we prove the Homotopy Lemma. On $M \times [0,1]$, if α is a k-form, $\alpha = \alpha_0 + dt \wedge \alpha_1$, where $\alpha_1 = i_{\partial_t}\alpha$ and $\alpha_0 = \alpha - dt \wedge i_{\partial_t}\alpha$.

$J_s : M \to M \times [0,1]$ such that $J_s(m) = (m,s)$. Hence $\partial_t(m,s) = \frac{d}{du}\Big|_{u=s} J_u(m)$.

Lemma 7.1 (Special case of Lie-Cartan formula).

$$\frac{d}{du}\Big|_{u=s}(J_u^*\alpha) = J_s^*(i_{\partial_t}d\alpha) + J_s^*d(i_{\partial_t}\alpha),$$

where $\alpha \in \Omega^k(M \times [0,1])$ and

$$J^*\alpha(m) : u \in [0,1] \mapsto (J_u^*\alpha)_m \in \bigwedge^k(T_mM).$$

$$\left(\frac{d}{du}\Big|_{u=s} J_U^*\alpha\right)_x := \frac{d}{du}\Big|_{u=s}[(J_u^*\alpha)_x].$$

Proof of Lemma. a) This is a local formula, hence we can prove it on $U \times [0,1]$ where U is the domain of a chart.

b) This is a linear formula.

Then it is enough to prove for

$$\alpha_0 = f(x,t)dt \wedge dx_1 \wedge \cdots \wedge dx_{q-1},$$

$$\alpha_1 = f(x,t)dx_1 \wedge \cdots \wedge dx_q,$$

where $(x_1, \cdots, x_n)$ are the coordinates in U.

First we consider α_1. $i_{\partial_t}\alpha_1 = 0$ and

$$d\alpha = \frac{\partial f}{\partial t}\, dt \wedge dx_1 \wedge \cdots \wedge dx_q + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_1 \wedge \cdots \wedge dx_q.$$

Thus, $i_{\partial_t} d\alpha = \frac{\partial f}{\partial t}\, dx_1 \wedge \cdots \wedge dx_n$. $J_s^*(i_{\partial_t} d\alpha) = \frac{\partial f}{\partial t}\, (m,s)dx_1 \wedge \cdots \wedge dx_q$.

$$J_u^*\alpha_1 = f(m,u)J_u^*dx_1 \wedge \cdots \wedge J_u^*dx_q = f(m,u)dx_1 \wedge \cdots \wedge dx_q.$$

Then $\frac{d}{du}\Big|_{u=s} J_u^*\alpha_1 = \frac{\partial f}{\partial t}\, (m,s)dx_1 \wedge \cdots \wedge dx_q$. Now the formula is proved for $\alpha = \alpha_1$.

For $\alpha = \alpha_0$, $i_{\partial_t}\alpha_0 = f dx_1 \wedge \cdots \wedge dx_{q-1}$, then

$$d(i_{\partial_t}\alpha_0) = \frac{\partial f}{\partial t}\, dt \wedge dx_1 \wedge \cdots \wedge dx_{q-1} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_1 \wedge \cdots \wedge dx_{q-1}.$$

$$\begin{aligned}
i_{\partial_t} d\alpha_0 &= i_{\partial_t}\left(\frac{\partial f}{\partial t}\, dt \wedge dt \wedge dx_1 \wedge \cdots \wedge dx_{q-1} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dt \wedge dx_1 \wedge \cdots \wedge dx_{q-1}\right) \\
&= i_{\partial_t}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dt \wedge dx_1 \wedge \cdots \wedge dx_{q-1}\right) \\
&= -\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_1 \wedge \cdots \wedge dx_{q-1}
\end{aligned}$$

Then

$$d(i_{\partial_t}\alpha_0) + i_{\partial_t} d\alpha_0 = \frac{\partial f}{\partial t}\, dt \wedge dx_1 \wedge \cdots \wedge dx_{q-1}.$$

$$J_s^*\left(d(i_{\partial_t}\alpha_0) + i_{\partial_t} d\alpha_0\right) = \frac{\partial f}{\partial t}\, (m,s)dx_1 \wedge \cdots \wedge dx_{q-1=0},$$

since $J_u^* dt = d(t \circ J_u) = 0$.

$J_u^*\alpha_0 = 0$. Thus the formula is also prove for the case $\alpha = \alpha_1$. $\square$

Set β a closed form on N and $F : M \times [0,1] \to N$ is the homotopy map of F_0 and F_1. Set $\alpha = F^*\beta$.

$$J_1^*\alpha = J_1^*F^*\beta = (F \circ J_1)^*\beta = F_1^*\beta.$$

$$J_0^*\alpha = J_0^*F^*\beta = (F \circ J_0)^*\beta = F_0^*\beta.$$

Then it suffices to show $J_1^*\alpha - J_0^*\alpha$ is exact.

Note that $d\alpha = dF^*\beta = F^*d\beta = 0$, and

$$\begin{aligned}
J_1^*\alpha - J_0^*\alpha &= \int_0^1 \left(\frac{d}{du}\Big|_{u=s} J_s^*\alpha\right) ds \\
&= \int_0^1 \left(J_s^*(i_{\partial_t} d\alpha) + J_s^* d(i_{\partial_t}\alpha)\right) ds \\
&= \int_0^1 \left(J_s^* d(i_{\partial_t}\alpha)\right) ds \\
&= d\int_0^1 \left(J_s^*(i_{\partial_t}\alpha)\right) ds
\end{aligned}$$

is exact. ☐

Remark 7.3. *From $d(\alpha+\beta) = d\alpha + d\beta$, we have $d\int = \int d$.*

Remark 7.4. *Lie-Cartan formula.*

$$L_\xi = d \circ i_\xi + i_\xi \circ d.$$

7.2 Cohomology group

A family of vector spaces associated to manifold

Definition 7.4. *Given $k \in \mathbb{N} > 0$, we define the k^{th} de Rham cohomology space*

$$H^k(M) := \{\omega \in \Omega^k(M) : d\omega = 0\}/\{\omega : \exists \alpha \ s.t. \ d\alpha = \omega\}.$$

We denote

$$\Omega_c^k(M) = \{\text{closed forms of degree } k\},$$

$$\Omega_e^k(M) = \{\text{exact forms of degree } k\}.$$

Then $\Omega_e^k(M) \subset \Omega_c^k(M)$ and $H^k(M) = \Omega_c(M)/\Omega_e(M)$.

Theorem 7.3 (De Rham)**.** *If M is compact, then $\dim(H^k(M)) < \infty$.*

Remark 7.5. *By M-V argument, we can prove the finiteness for a manifold with finite good cover, see the book by Bott and Tu.*

Definition 7.5. *The k^{th} Betti number of M is*

$$b^k(M) := \dim(H^k(M)).$$

Goal of this section is to show that $b^i(\mathbb{S}^n) = \begin{cases} 1, & \text{if } i = n, 0 \\ 0, & \text{otherwise} \end{cases}$.

Note that $b^k(M) = 0$ is equivalent to every closed k-form is exact.

For $\omega \in \Omega^k(M)$, if ω is closed, then it is in Ω_c^k, and we use $[\omega]$ to denote the cohomology class in $H^k(M)$ respect to ω.

Proposition 7.2. *Show that if smooth $F : M \to N$ and $[\alpha] = [\beta]$ for $\alpha, \beta \in \Omega^k(N)$, then $[F^*\alpha] = [F^*\beta]$.*

Proof. By definition, there is $\omega \in \Omega^{k-1}(N)$ such that $\alpha = \beta + d\omega$, then

$$\begin{aligned} F^*\alpha &= F^*(\beta + d\omega) \\ &= F^*\beta + F^*d\omega \\ &= F^*\beta + dF^*\omega, \end{aligned}$$

hence $[F^*\alpha] = [F^*\beta]$. ☐

Proposition 7.3. *If $F : M \to N$ and $G : M \to N$ are homotopic. For any closed form $\alpha \in \Omega_c^k(N)$, we have $[F^*\alpha] = [G^*\alpha]$.*

Proof. This is just the Homotopy Lemma 7.2. $\square$

Definition 7.6. *If $F : M \to N$, we define*

$$\begin{aligned} F^* : H^k(N) &\to H^k(M), \\ \omega &\mapsto F^*\omega = [F^*\alpha], \ \textit{if } [\alpha] = \omega. \end{aligned}$$

It is well-defined due to Proposition 7.2.

Theorem 7.4 (Homotopy Lemma)**.** *If F and G are homotopic $M \to N$, then $F^* = G^* : H^k(N) \to H^k(M)$.*

Definition 7.7. *We say $F : M \to N$ is a* ***homotopy equivalence****, if there is $G : N \to M$ such that $F \circ G \sim \mathrm{id}_N$ and $G \circ F \sim \mathrm{id}_M$.*

Proposition 7.4. *If F is a homotopy equivalence between M and N, then $b^k(M) = b^k(N)$.*

Proof. If $F \circ G \sim \mathrm{id}$, by homotopy lemma $\mathrm{id} = (\mathrm{id})^* = (F \circ G)^* = G^* \circ F^*$. Similarly, we have $F^* \circ G^* = \mathrm{id}$. Hence F^* is a bijection between $H^k(N)$ and $H^k(M)$, with the inverse G^*, which indicates

$$b^k(M) = b^k(N).$$

$\square$

Proposition 7.5. *Show that $M \times \mathbb{R}$ is homotopy equivalent to M.*

Proof. We will construct $F : M \times \mathbb{R} \to M$ which is a homotopy equivalence. Define

$$\begin{aligned} F : M \times \mathbb{R} &\to M, \\ (m, t) &\mapsto m. \end{aligned}$$

$$\begin{aligned} G : M &\to M \times \mathbb{R}, \\ m &\mapsto (m, 0). \end{aligned}$$

Then

$$F \circ G = \mathrm{id}, \text{ and } G \circ F : (m, t) \mapsto (m, 0).$$

We can have

$$\begin{aligned} H : M \times \mathbb{R} \times [0, 1] &\to M \times \mathbb{R}, \\ \big((m, t), s\big) &\mapsto (m, st). \end{aligned}$$

Note that H is smooth, $H\big((m, t), 0\big) = (m, 0) = G \circ F(m, t)$ and $H\big((m, t), 1\big) = (m, t) = \mathrm{id}_{M\times\mathbb{R}}$. Hence $G \circ F \sim \mathrm{id}_{M\times\mathbb{R}}$. $\square$

Corollary 7.1. $b^k(M \times \mathbb{R}) = b^k(M)$.

Corollary 7.2. $b^k(\mathbb{R}^2\backslash\{0\}) = b^k(\mathbb{S}^1)$.

7.3 Cohomology of Spheres

Remark 7.6. *Set M is a manifold with $\dim M = m$. Then $\Omega^{k+1}(M) = 0$ and hence $H^{k+1}(M) = 0$ for $k \geq m$.*

$H^0(M) = \{\omega \in \Omega_c^0(M)\}/\{exact\ forms\} = \{f : df = 0\}/\{0\}$, i.e., $H^0(M)$ is the set of locally constant functions, hence $b^0(M)$ is the number of connected components of M. Thus $b^0(M) = 1$ if M is connected.

Exercise 7.1 (Mayer Vietoris)**.** *$u = (1, 0, \cdots, 0) \in \mathbb{S}^n$ and $v = (-1, 0, \cdots, 0) \in \mathbb{S}^n$. Define two open sets $U = \mathbb{S}^n \backslash \{u\}$ and $V = \mathbb{S}^n \backslash \{v\}$. Show that*

(i) U and V are contractible.

(ii) $U \cap V$ is homotopic equivalent to $\mathbb{S}^{n-1}$

Proof.

(i) Define $K_v : U \to \{v\}$ by $K_v(x) = v$. We define

$$\begin{aligned} H : U \times [0,1] &\to \mathbb{S}^n, \\ (x,t) &\mapsto \frac{(1-t)v + tx}{\|(1-t)v + tx\|}. \end{aligned}$$

In fact, $H(x, 1) = x = \mathrm{id}(x)$ and $H(x, 0) = v = K_v(x)$. Obvious H is smooth, we only have to verify that H is well-defined, that is, $\|(1-t)v + tx\| \neq 0$.

For $x \neq v$, since $x \neq u = -v$, x and v are independent, if $(1-t)v + tx = 0$, we have $(1-t) = t = 0$, which is impossible. For $x = v$, $\|(1-t)v + tx\| = \|v\| = 1$.

(ii) We treat $\mathbb{S}^{n-1}$ as a submanifold of $\mathbb{S}^n$ and the inclusion map is

$$i : \mathbb{S}^{n-1} \to \mathbb{S}^n, \quad i(x_1, \cdots, x_n) = (0, x_1, \cdots, x_n).$$

Now we define

$$r : \mathbb{S}^n \to \mathbb{S}^{n-1}, \quad r(x_0, x_1, \cdots, x_n) = \frac{(x_1, \cdots, x_n)}{x_1^2 + \cdots + x_n^2}.$$

Then

$$\begin{aligned} i \circ r(x_0, \cdots, x_n) &= \left(0, \frac{x_1}{x_1^2 + \cdots + x_n^2}, \cdots, \frac{x_n}{x_1^2 + \cdots + x_n^2}\right) \\ r \circ i(x_1, \cdots, x_n) &= (x_1, \cdots, x_n) = \mathrm{id}_{\mathbb{S}^{n-1}}(x_1, \cdots, x_n). \end{aligned}$$

Thus we only need to show that $i \circ r \sim \mathrm{id}_{\mathbb{S}^n}$.

$H : U \cap V \to \mathbb{S}^n$ by $H\big((x_0, x_1, \cdots, x_n), t\big) = \frac{(tx_0, x_1, \cdots, x_n)}{\|(tx_0, x_1, \cdots, x_n)\|}$. Then

$$H\big((x_0, x_1, \cdots, x_n), 0\big) = \frac{(0, x_1, \cdots, x_n)}{\|(0, x_1, \cdots, x_n)\|} = \left(0, \frac{x_1}{x_1^2 + \cdots + x_n^2}, \cdots, \frac{x_n}{x_1^2 + \cdots + x_n^2}\right) = i \circ r(x_0, \cdots, x_n).$$

$$H\big((x_0, x_1, \cdots, x_n), 1\big) = (x_0, \cdots, x_n) = \mathrm{id}_{\mathbb{S}^n}(x_0, \cdots, x_n).$$

□

Lemma 7.2. *There is a unique (linear) map*

$$J : H^k(\mathbb{S}^n) \to H^{k-1}(U \cap V),$$
$$[\omega] \mapsto [\alpha - \beta],$$

where $\alpha \in \Omega^{k-1}(U)$ with $d\alpha = \omega|_U$ and $\beta \in \Omega^{k-1}(V)$ with $d\beta = \omega|_V$..

Proof. First we explain the existence of α and β. Since $\omega|_U \in \Omega^k(U)$ and $d\omega|_U = (d\omega)|_U = 0$, and U is contractible, then we have $\alpha \in \Omega^{k-1}(U)$ such that $d\alpha = \omega|_U$. Ditto for β.

Then we prove that J is well-defined. First we prove it doesn't depend on the choice of α and β, then we prove it doesn't depend on the choice of the representation of $[\omega]$.

If α and α' are in $\Omega^{k-1}(U)$ and $d\alpha = d\alpha' = \omega|_U$, then $d(\alpha - \alpha') = 0$. Since U is contractible, there is a form $\gamma \in \Omega^{k-2}(U)$ such that $\alpha = \alpha' + d\gamma$. Similarly, if β and β' are in $\Omega^{k-1}(V)$ and $d\beta = d\beta' = \omega|_V$, then there is a form $\eta \in \Omega^{k-2}(V)$ such that $\beta = \beta' + d\eta$. Now we have

$$\alpha - \beta = \alpha' - \beta' + d(\gamma - \eta),$$

which indicates $[\alpha - \beta] = [\alpha' - \beta'] \in H^{k-1}(U \cap V)$.

Second, if $[\omega] = [\omega'] \in H^k(\mathbb{S}^n)$, i.e. there is a form $\theta \in \Omega^{k-1}(\mathbb{S}^n)$ such that $\omega = \omega' + d\theta$. Taking $\alpha' = \alpha - \theta|_U$ and $\beta' = \beta - \theta|_V$, then we have

$$d\alpha' = d\alpha - d\theta|_U = \omega|_U - d\theta|_U = \omega'|_U,$$

$$d\beta' = d\beta - d\theta|_V = \omega|_V - d\theta|_V = \omega'|_V,$$

$$(\alpha' - \beta')|_{U\cap V} = \alpha|_{U\cap V} - \theta|_{U\cap V} - \beta|_{U\cap V} + \theta|_{U\cap V} = \alpha|_{U\cap V} - \beta|_{U\cap V} = (\alpha - \beta)|_{U\cap V}.$$

□

Theorem 7.5. *$J : H^k(\mathbb{S}^n) \to H^{k-1}(U \cap V)$ is bijective when $k - 1 > 0$ and $b^k(\mathbb{S}^n) = b^{k-1}(U \cap V) = b^{k-1}(\mathbb{S}^{n-1})$.*

For example, $b^2(\mathbb{S}^3) = b^1(\mathbb{S}^2)$.

Idea: we use a function ψ on $\mathbb{S}^n$ such that

$$\begin{cases} \psi = 1 \text{ on a neighborhood of } u, \\ \psi = 0 \text{ on a neighborhood of } v. \end{cases}$$

If α is a form defined on $U = \mathbb{S}^n \backslash \{v\}$, then

$$\psi \cdot \alpha = \begin{cases} \psi \cdot \alpha \text{ on } \ U, \\ 0 \text{ on a neighborhood of } v. \end{cases}$$

is a global smooth form on $\mathbb{S}^n$.

Likewise, if β is a form defined on V, then $(1 - \psi)\beta$ is defined on $\mathbb{S}^n$.

Proof. First we show that J is injective, that is, if $J[\omega] = 0$, then $[\omega] = 0$. Now we have $\alpha \in \Omega^{k-1}(U)$, $\beta \in \Omega^{k-1}(V)$, $d\alpha = \omega|_U$ and $d\beta = \omega|_V$.

$$0 = J[\omega] = [\alpha - \beta],$$

indicates that there is a form $\gamma \in \Omega^{k-2}(U \cap V)$ such that $\alpha = \beta + d\gamma$. Then we will construct $\eta \in \Omega^{k-1}(\mathbb{S}^n)$ such that $\omega = d\eta$.

Note that γ is defined on $U \cap V$, now we will use ψ to construct forms on U and V. $\psi\gamma$ is well-defined on V (there is a gap in U) and $(1-\psi)\gamma$ is well-defined on U. Then we define

$$\tilde{\alpha} = \alpha - d((1-\psi)\gamma,$$
$$\tilde{\beta} = \beta + d(\psi\gamma).$$

Then on $U \cap V$,

$$\tilde{\alpha} - \tilde{\beta} = \alpha - \beta - d((1-\psi)\gamma + \psi\gamma) = \alpha - \beta - d\gamma = 0,$$

which means $\tilde{\alpha} = \tilde{\beta}$ on $U \cap V$. Now we define

$$\eta = \begin{cases} \tilde{\alpha}, & \text{on } U \\ \tilde{\beta}, & \text{on } V. \end{cases}$$

Since $\tilde{\alpha} = \tilde{\beta}$, η is well-defined on $\mathbb{S}^n$.

Note that on U, we have

$$\omega|_U = d\alpha = d\tilde{\alpha} = d(\eta|_U),$$

and on V we have

$$\omega|V = d\beta = d\tilde{\beta} = d(\eta|_V),$$

that is $\omega = d\eta$, i.e. $[\omega] = 0 \in H^k(\mathbb{S}^n)$.

It remains to prove that J is surjective. For any form in $H^{k-1}(U \cap V)$, we choose $\gamma \in \Omega_c^{k-1}(U \cap V)$ to represent it. Now we need to find $\omega \in \Omega_c^k(\mathbb{S}^n)$, $\alpha \in H^{k-1}(U)$ and $\beta \in H^{k-1}(V)$ such that

$$d\alpha = \omega|_U, \quad d\beta = \omega|_V, \quad [\alpha - \beta] = [\gamma].$$

Since $d\psi = 0$ on $V(u)$ and $V(v)$, $d\psi \wedge \gamma$ is well-defined on $U \cup V = \mathbb{S}^n$. Let show that $J[-d\psi \wedge \gamma] = [\gamma]$. Define

$$\alpha := (1-\psi)\gamma, \quad d\alpha = -d\psi \wedge \gamma, \text{on } U.$$

$$\beta := \alpha - \gamma = -\psi\gamma, \quad d\beta = -d\psi \wedge \gamma, \text{on } V.$$

□

Proposition 7.6. $b^1(\mathbb{S}^n) = \begin{cases} 0, & \textit{if } n > 1, \\ 1, & \textit{if } n = 1. \end{cases}$

Proof. If α is a closed 1-form on $\mathbb{S}^n$, then there are functions f and g defined on U and V such that $df = \alpha|_U$ and $dg = \alpha|_V$. Then $d(f-g) = 0$ on $U \cap V$.

If $n > 1$, $U \cap V$ is connected, hence $f - g$ is a constant on $U \cap V$, denoted by $\lambda \in \mathbb{R}$. Then we define

$$h := \begin{cases} f, & \text{on } U \\ g + \lambda, & \text{on} V. \end{cases}$$

Then $\alpha = dh$.

If $n = 1$, note that $U \cap V = O_1 \sqcup O_2$. Then $f - g = \lambda_1$ on O_1 and $f - g = \lambda_2$ on O_2. Now we define

$$J : H^1(\mathbb{S}^1) \to H^0(\mathbb{S}^0) = \mathbb{R}, \quad [\alpha] \mapsto \lambda_1 - \lambda_2$$

is a bijection. (This proof is similar to the proof of the last theorem).

First we say that J is well-defined. For $[\alpha'] = [\alpha]$, there is a 0-form l such that $\alpha' = \alpha + dl$, then we define $f' = f + l$, and $g' = g + l$, then

$$df' = df + dl = \alpha|_U + dl = \alpha'|_U,$$

$$dg' = dg + dl = \alpha|_V + dl = \alpha'|_V.$$

And $f' - g' = f - g$. So λ_1 and λ_2 keep invariant and hence so does $\lambda_1 - \lambda_2$.

It's not difficult to show that J is injective. If $\lambda_1 - \lambda_2 = 0$, then similar to the case for $n > 1$, we can construct a global function h such that $\alpha = dh$.

Now we will prove that J is surjective. For any $c \in \mathbb{R}$, we construct a function g such that g is 0 around u^- and g is c around u^+.

We assume O_1 contains a neighborhood of u^- and O_2 contains a neighborhood of u^+. Now we define

$$f := \begin{cases} g + c, & \text{on } O_1, \\ g, & \text{on } O_2. \end{cases}$$

Then we have $f - g = c$ on O_1 and $f - g = 0$ on O_2, that is, $\gamma_1 = c$ and $\gamma_2 = 0$. We need to verify f is well-defined, that is, f is smooth at point u. Since $f = g + c = c$ around u^- and $f = g = c$ round u^+, we say f is continuous at point u. Moreover, g is constant around u^- or u^+, then f is smooth at point u. □

Remark 7.7. *For calculating $H^1(\mathbb{S}^1)$, we can also define*

$$\int : H^1(\mathbb{S}^1) \to \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{S}^1} \omega.$$

We will show that $\int$ is a bijection.

If $\int_{\mathbb{S}^1} \omega = 0$, then

$$\int_{u^-}^{u^+} \omega = \int_{\mathbb{S}^1 \setminus \{u\}} \omega = 0.$$

Since on $\mathbb{S}^1\backslash\{u\}$*, there is a function* f *such that* $\omega = df$*, then*

$$0 = \int_{u^-}^{u^+} df = f(u^+) - f(u^-).$$

Now we define $f(u) = f(u^+) = f(u^-)$*. then we have a global function on* $\mathbb{S}^1$*. But we need to show* f *is smooth. However, the smoothness is due to* $df = \omega$ *on* $\mathbb{S}^1\backslash\{u\}$ *and the smoothness of* ω*.*

It suffices to prove $\int$ *is surjective. For any* $c \in \mathbb{R}$*,* $\omega = \frac{c}{2\pi} d\theta$ *is just what we need.*

Corollary 7.3. $b^i(\mathbb{S}^n) = \begin{cases} 1, & \textit{if } i = n, 0 \\ 0, & \textit{otherwise} \end{cases}.$

8 Orientation and Manifold with boundary

8.1 Orientation

Definition 8.1. *On a manifold of dimension n, a **volume form** is a form ω of degree n, such that $\omega_x \neq 0$ for every x in M.*

Remark 8.1. *Recall $\bigwedge^n(E^*) = 1$, where $\dim E = n$ a vector space. Then a basis of $\bigwedge^n(E^*)$ is given by $e_1 \wedge \cdots \wedge e_n$ where $(e^1, \cdots, e^n)$ is a basis of E with $\dim(\bigwedge(E^*)) = 1$.*

Example 8.1. *$\omega = dx_1 \wedge \cdots \wedge dx_n$ is a volume form on $\mathbb{R}^n$.*

$\varphi : \mathbb{R}^n \to \mathbb{R}^n$ a smooth map, then

$$\varphi^*(\omega) = \mathrm{Jac}(\varphi)\omega,$$

where $\mathrm{Jac}(\varphi)$ is the function defined on $\mathbb{R}^n$ by

$$\mathrm{Jac}(\varphi)(x) = \det(D_x\varphi) = \det\left(\frac{\partial \varphi_i}{\partial x_j}\right).$$

$$\varphi^*(\omega)_x(u_1, \cdots, u_n) = \omega_{\varphi(x)}(D_x\varphi(u_1), \cdots, D_x\varphi(u_n)) = \det(D_x\varphi)\omega_x(u_1, \cdots, u_n).$$

Definition 8.2. *A manifold M is **orientable** if there exists a volume form on M.*

Exercise 8.1. *We will prove $\mathbb{S}^n/\{-\mathrm{id}\} = \mathbb{RP}^n$ is orientable iff n is odd ("impair").*

Remark 8.2. *Assume now that M is connected, then if we choose a volume form ω_0 on M, then every form of degree $n = \dim M$, then every ω in $\Omega^n(M)$ is of the form $\omega = f\omega_0$, where $f \in C^\infty(M)$.*

In particular, if ω is a volume form, f never vanishes (nowhere 0). Thus if M is connected, either $f > 0$, nor $f < 0$.

Definition 8.3. *We have the following equivalence relation, two volume form ω_1 and ω_2 on M, defines the same orientation if and only if*

$$\omega_1 = f\omega_2,$$

with $f > 0$.

Exercise 8.2. *Show that this indeed is an equivalence relation.*

Definition 8.4. *An **orientation** on M is the choice of a class in the above equivalence relation, that is a choice of a volume form, up to multiplication by a positive function.*

If M is connected and orientable, M has two orientation, one given by ω_0 and another by $-\omega_0$.

*M is **oriented** if it is orientable and an orientation has been chosen.*

Definition 8.5. *$\varphi : M \to N$ is a diffeomorphism, M is oriented by ω_1, N is oriented by ω_1. We say φ **preserves the orientation** if $\varphi^*\omega_1$ defines the same orientation as ω_0.*

Example: $M, N = \mathbb{R}^n$, φ preserves the orientation, iff $\mathrm{Jac}(\varphi) > 0$.

Remark 8.3. *If φ does not preserves the orientation for M, N connected, then φ* ***reverses the orientation****, $\varphi^*\omega_1 \sim \varphi_0$.*

Theorem 8.1. *M is orientable, iff there exists an atlas (V_i, φ_i) on M that that* $\det\big(\operatorname{Jac}(\varphi_i \circ \varphi_j^{-1})\big) > 0$ *on $U_i \cap U_j$.*

Proof. We can always assume for simplicity that M is connected.

Assume M is oriented by ω_0. Let $(U_i, \tilde{\varphi}_i)$ be an atlas on M, where U_i is connected.

If $\tilde{\varphi}_i : U_i \subset M \to \mathscr{O}_i \subset \mathbb{R}^n$ preserves the orientation. Then we take $\varphi_i = \tilde{\varphi}_i$.

If $\tilde{\varphi}_i : U_i \subset M \to \mathscr{O}_i \subset \mathbb{R}^n$ reverses the orientation. Then we take $\varphi_i = A \circ \tilde{\varphi}_i$, where A is a linear map with $\det A = -1$.

Then φ_i preserves the orientation,

$$\begin{aligned}(\varphi_i \circ \varphi_j^{-1})^*\omega &= (\varphi_j^{-1})^*(\varphi_i^*\omega)\\ &= (\varphi_j^{-1})^*(f_i\omega_0)\\ &= (f_i \circ \varphi_j^{-1})((\varphi_j^{-1})^*\omega_0)\\ &= (f_i \circ \varphi_j^{-1})g_j\omega,\end{aligned}$$

where f and g are positive function and ω is the volume form $dx_1 \wedge \cdots \wedge dx_n$ in $\mathbb{R}^n$. Hence $\det\big(\operatorname{Jac}(\varphi_i \circ \varphi_j^{-1})\big) > 0$.

Assume that $\det\big(\operatorname{Jac}(\varphi_i \circ \varphi_j^{-1})\big) > 0$. Let ω_i on U_i with $\omega_i = \varphi_i^*\omega$.

On $U_i \cap U_j$, $\omega_i = g_{ij}\omega_j$, where g_{ij} is a function on $U_i \cap U_j$. The hypothesis gives that $g_{ij} > 0$, indeed,

$$\begin{aligned}\omega &= (\varphi_i^{01})^*\big(g_{ij}\varphi_j^*(\omega)\big)\\ &= (g_{ij} \circ \varphi_i^{-1})\big((\varphi_j \circ \varphi_i^{-1})^*\omega\big)\\ &= (g_{ij} \circ \varphi_i^{-1}) \det\big(\operatorname{Jac}(\varphi_j \circ \varphi_i^{-1})\big)\omega.\end{aligned}$$

Let ψ_i be a partition of unity associated to U_i (we also assume $\{U_i\}$ is locally finite). Then $\operatorname{Supp}(\psi_i) \subset U_i$, $\sum \psi_i = 1$ and $\psi_i \geq 0$. We take $\omega_0 = \sum_i \psi_i\omega_i$. Let us finally prove $(\omega_0)_x \neq 0$ for any $x \in M$.

Let i_0 such that $\psi_{i_0}(x) > 0$. $\omega_0(x) = \sum_{i:x\in U_i} \psi_i(x)\omega_i(x)$, where the summation is finite.

$$\begin{aligned}\omega_0(x) &= \psi_{i_0}(x)\omega_{i_0}(x) + \sum_{i\neq i_0:x\in U_i} \psi_i(x)g_{ii_0}(x)\omega_0(x)\\ &= \Big(\psi_{i_0}(x) + \sum_{i\neq i_0:x\in U_i} \psi_i(x)g_{ii_0}(x)\Big)\omega_{i_0}(x),\end{aligned}$$

where $\psi_i(x)g_{ii_0}(x) \geq 0$ and $\psi_{i_0}(x) > 0$. $\square$

Given an oriented manifold $M<$ we can define $\int_M \omega$, where $\omega \in \Omega^n(M)$, $n = \dim M$.

8.2 Manifold with boundary

Model:

(i) Half space $\mathbb{H}^n = \{(x_1, \cdots, x_n) : x_1 \leq 0\}$.

(ii)Boundary $\partial\mathbb{H}^n = \{(x_1, \cdots, x_n) : x_1 = 0\}$.

Remark 8.4. *The boundary of U in topology language is $\overline{U} \setminus U$.*

(iii)For U an open set in $\mathbb{H}^n$, the boundary of U is $\partial U = U \cap \partial\mathbb{H}^n$.

(iv) A function (or mapping) continuous $f : U \to \mathbb{R}$ or $\mathbb{R}^n$ is smooth if there exists a smooth g defined on $\mathscr{O} \supset U$ open set of $\mathbb{R}^n$ such that $g|_U = f$.

(v) $f : U \subset \mathbb{H}^n \to V \subset \mathbb{H}^n$ is a diffeomorphism, if F is smooth, bijective and the inverse f^{-1} is smooth.

Proposition 8.1. *If f is a diffeomorphism from $U \subset \mathbb{H}^n$ to $V \subset \mathbb{H}^n$, then $f(\partial U) = \partial V$.*

Manifold with boundary. M a nice topological space.

Define chart (U, X) where X bijection from U to an open set in $\mathbb{H}^n$. (U, X) and (V, Y) are C^∞ compatible if $X \circ Y^{-1}$ and $Y \circ X^{-1}$ are smooth.

Alas on M gives the definition of manifold with boundary. $x \in M$ belongs to the boundary of M, if there exists a chart (U, φ), $x \in U$, $\varphi(x) \in \partial(\varphi(U))$. (The definition is not depend on the choice of the chart, due to the last proposition.)

Proposition 8.2. *If φ is a diffeomorphism from M to N, then $\varphi(\partial M) = \partial N$.*

∂M is a submanifold of M, $\dim \partial M = \dim M - 1$.

Exercise 8.3. *$M \setminus \partial M$ is a usual manifold.*

The question is how to define the vector space of a boundary point $m \in \partial M$.

$T_m^*\mathbb{H}^n = \{\text{functions on } \mathbb{H}^n\}/\{d_m f = 0\} = \{\text{functions on } \mathbb{R}^n\}/\{d_m f = 0\} = T_m^*\mathbb{R}^n$.

Definition 8.6. *Let $v \in T_m M$, $m \in \partial M$. We say v is* ***tangent to the boundary*** *if $v \in T_m \partial M$. We say v is* ***outward normal*** *if v is not tangent to the boundary and there exists $c : [0, 1] \to M$ such that $\dot{c} = v$.*

Proposition 8.3. *Assume v, w are outward normal at m, then $v = \lambda w + u$, where $\lambda > 0$ and $u \in T_m \partial M$.*

Proof. It is enough to prove it in a chart that is for $M = \mathbb{H}^n$. □

Proposition 8.4. *Given M there exists a vector field ξ along ∂M, such that for any $x \in \partial M$, $\xi(x)$ is outward normal.*

Proof. It is true on $\mathbb{H}^n$, $\xi = \frac{\partial}{\partial x_1}$.

Take an atlas (U_i, φ_i) on M, locally finite. On $U_i \cap \partial M$, define $\xi_i = (\varphi_i^{-1})_x(\frac{\partial}{\partial x_1})$.

Take ψ_i a partition of unity, then we define $\xi = \sum \psi_i \cdot \xi_i$. □

Definition 8.7. *Assume M is oriented, then the* **canonical orientation** *of ∂M is given by the form $\omega_1 = i_\xi \omega$, where ξ is an outward normal.*

The orientation on $\partial \mathbb{H}^n$ is given by $dx_2 \wedge \cdots \wedge dx_n$.

Remark 8.5. *U open set in $\mathbb{R}^n$, then $\overline{U}$ is a manifold with boundary and $\partial \overline{U} = \mathrm{Fr}(U) := \overline{U} \setminus U$.*

8.3 More on differential forms

Exercise 8.4. *For $X = \sum f_i \frac{\partial}{\partial x_i}$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$, what is $d i_X \omega$?*

Proof.

$$i_X \omega = \sum (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n.$$

$$d i_X \omega = \sum \frac{\partial f_i}{\partial x_i} \omega.$$

□

Exercise 8.5. *Find a volume form of $\mathbb{S}^n$ to make it orientable.*

Proof. Set $\omega = dx_0 \wedge \cdots \wedge dx_n$. For any point $x = (x_0, \cdots, x_n) \in \mathbb{S}^n$ and then $X = \sum x_i \frac{\partial}{\partial x_i}$ is the normal vector at x, hence we take

$$i_X \omega = \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n.$$

□

Exercise 8.6. *$\psi : \mathbb{S}^n \to \mathbb{S}^n$ takes u to $-u$, prove that ψ preserves the orientation if n is odd, and reverses the orientation if n is even.*

Proof. Just prove that $\psi^*(i_X \omega) = (-1)^{n+1} i_X \omega$. □

Exercise 8.7. $\mathbb{S}^n \xrightarrow{p} \mathbb{RP}^n \approx \mathbb{S}^n / \{\pm \mathrm{id}\}$. *So $p \circ \psi = p$.*

Show that if n is even, then $\mathbb{RP}^n$ is not oriented.

Show that if n is odd, then $\mathbb{RP}^n$ is oriented.

Proof. Argue by contradiction for n even. Suppose $\mathbb{RP}^n$ is orientable, then there is a volume form ω_0 on $\mathbb{RP}^n$. Hence $p^* \omega_0$ is a volume form on $\mathbb{S}^n$ since p is a local diffeomorphism.

Note that $p \circ \psi = p$ indicates $p^* = \psi^* p^*$, hence $p^* \omega_0 = \psi^*(p^* \omega_0)$, which shows that ψ preserves the orientation. That's a contradiction due to the last exercise.

For n odd. By local diffeomorphism of p, we can push forward the volume form on $\mathbb{S}^n$. Notice that we should prove that the definition of volume form on $\mathbb{RP}^n$ does not depend on the choice of quotient map p or $p \circ \psi$. □

9 Integration of Differential Forms on Oriented Manifold

Goal: $\omega \in \Omega^n(M)$ with compact support and M has dimension n with M is oriented. We want to define $\int_M \omega$.

Remark 9.1. *Some people says $\int_M \omega = 0$ if ω is not of degree equals to the dimension of M.*

9.1 On $\mathbb{R}^n$

Now $\omega = f dx_1 \wedge \cdots \wedge dx_n$. We define

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f dx_1 \cdots dx_n,$$

here $dx_1 \cdots dx_n$ is the Lebesgue measure on $\mathbb{R}^n$.

Change of variable formula, set φ a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$, then

$$\int_U (f \circ \varphi) |\det J(\varphi)| dx_1 \cdots dx_n = \int_{\varphi(U)} f dx_1 \cdots dx_n.$$

Assume ω is supported in an open set $U \subset \mathbb{R}^n$, assume that φ preserves the orientation, φ a diffeomorphism from U to $\varphi(U)$,

$$\int_{\varphi(U)} \omega = \int_U \varphi^* \omega.$$

9.2 On Manifold

Suppose ω has compact support in a domain U of a chart $U \subset M$ and the coordinates map is φ.

Proposition 9.1. *If ω has compact support in (U, φ) and (V, ψ) where φ and ψ preserve the orientation, then*

$$\int_{\varphi(U)} (\varphi^{-1})^* \omega = \int_{\psi(U)} (\psi^{-1})^* \omega.$$

Definition 9.1. *Let (U_i, φ_i) be an atlas of M, where φ_i preserves the orientation. Let ψ_i be a partition of unity subordinated to U_i.*

$$\int_M^{(U_i, \varphi_i, \psi_i)} \omega = \sum_{i \in I} \int_{\varphi_i(U_i)} [(\varphi_i^{-1})^* (\psi_i \omega)].$$

Proposition 9.2. *$\int_M^{(U_i, \varphi_i, \psi_i)} \omega$ does not depend on the choice of (U_i, φ_i, ψ_i) and then we define*

$$\int_M \varphi := \int_M^{(U_i, \varphi_i, \psi_i)} \omega.$$

Proof.

$$\begin{aligned}
\int_M^{(\overline{U}_j,\overline{\varphi}_j,\overline{\psi}_j)} \omega &= \sum_j \int_{\overline{\varphi}_j(\overline{U}_j)} \left(\overline{\varphi}_j^{-1}\right)^* (\overline{\psi}_j \omega) \\
&= \sum_{i,j} \int_{\overline{\varphi}_j(\overline{U}_j)} \left(\overline{\varphi}_j^{-1}\right)^* (\overline{\psi}_j \psi_i \omega) \\
&= \sum_{i,j} \int_{\overline{\varphi}_j(\overline{U}_j \cap U_i)} \left(\overline{\varphi}_j^{-1}\right)^* (\overline{\psi}_j \psi_i \omega) \\
&= \sum_{i,j} \int_{\varphi_i(\overline{U}_j \cap U_i)} \left(\varphi_i^{-1}\right)^* (\overline{\psi}_j \psi_i \omega) \\
&= \sum_{i} \int_{\varphi_i(\overline{U}_j \cap U_i)} \left(\varphi_i^{-1}\right)^* (\psi_i \omega) \\
&= \int_M^{(U_i,\varphi_i,\psi_i)} \omega.
\end{aligned}$$

(Partition twice.) ☐

Proposition 9.3. *$\overline{M}$ is M with the opposite orientation,*

$$\int_M \omega = -\int_{\overline{M}} \omega.$$

Proposition 9.4. *If $\varphi : M \to N$ is diffeomorphism preserving the orientation, then*

$$\int_{\varphi(M)} \omega = \int_M \varphi^* \omega.$$

9.3 Stokes Formula

Let M be an oriented manifold with boundary ∂M. ∂M is an oriented manifold with orientation $i_\xi \omega$ where ξ is an outward vector field and ω defining the orientation on M.

Theorem 9.1. *For $\alpha \in \Omega^n(M)$ with compact support and $\dim M = n$,*

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

Exercise 9.1. *Show that*

$$\int_{[a,b]} df = \int_b f + \int_{\overline{a}} f = \int_{\partial[a,b]} f.$$

Proof. $\omega = dx_1$. $i_{dx_1}\omega = 1$ at b, and $i_{-dx_1}\omega = -1$. ☐

Proof. This formula is linear in α, then it is enough to prove it for α with support in a chart. By $\mathbb{H}^n$ we mean $\{(x_1 \leq 0, x_2, \cdots, x_n)\}$

$$\int_P \alpha = \int_{\mathbb{H}^n} d\alpha.$$

$$\alpha = f dx_2 \wedge \cdots \wedge dx_n + dx_1 \wedge \sum_{i=2}^n g_i dx_2 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_n.$$

Since $x_1 \equiv 0$ on P, then all the terms containing dx_1 vanishes. Then

$$\int_P = \int_{\mathbb{R}^{n-1}} f(0, x_2, \cdots, x_n) dx_2 \wedge \cdots \wedge dx_n.$$

Now $d\alpha = \frac{\partial f}{\partial x_1}\, dx_1 \wedge \cdots \wedge dx_n + \sum_{i=2}^{n} (-1)^i \frac{\partial g_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n$, then

$$\int_{\mathbb{R}^{n-1}} \left(\int_{x_1 \le 0} \left(\frac{\partial f}{\partial x_1}\, dx_1 \right) \right) dx_2 \cdots dx_m = \int_{\mathbb{R}^{n-1}} \left(f(0, x_2, \cdots, x_n) \right) dx_2 \cdots dx_m.$$

$$\int_{\mathbb{R}^{n-1}} \left(\int_{x_i \in \mathbb{R}} \frac{\partial g_i}{\partial x_i}\, dx_i \right) dx_1 \cdots \hat{dx_i} \cdots dx_n = 0.$$

Here we used the fact ω is compactly supported. □

Corollary 9.1. *If M has no boundary, then*

$$\int_M d\omega = 0.$$

Theorem 9.2 (Brouwer fixed point theorem)**.** *Let $\varphi : B^n \to B^n$ smooth (continuous), there is $x \in B$ such that $\varphi(x) = x$.*

Definition 9.2. *A **retraction** is a smooth map*

$$F : B^n \to \mathbb{S}^{n-1} = \partial B_n$$

such that $F|_{\mathbb{S}^{n-1}} = \mathrm{id}$.

Proposition 9.5. *There is no retraction from B^n to $\mathbb{S}^{n-1}$.*

Proof. Let F be a retraction from B^n to $\mathbb{S}^{n-1}$. Let ω be the volume form on $\mathbb{S}^{n-1}$, then

$$0 \neq \int_{\mathbb{S}^{n-1}} \omega = \int_{\mathbb{S}^{n-1}} F^*\omega = \int_{B^n} dF^*\omega = \int_{B^n} F^*(d\omega) = 0.$$

□

Proof of Brower fixed point theorem. For $x \in B^n$ and $\varphi(x) \neq x \in B^n$, consider the directed line $\varphi(x)x$ intersecting $\mathbb{S}^{n-1}$ at $F(x)$, hence F is the retraction. □

Proposition 9.6. *If $\omega = f\omega_0$, where ω_0 is the orientation. If $f \geq 0$ and there is $m \in M$ such that $f(m) > 0$, then $\int \omega > 0$.*

We will prove it for M oriented and closed (no boundary and compact), the following map is an isometry

$$H^n(M) \to \mathbb{R}$$
$$\omega \mapsto \int_M \omega$$

Exercise 9.2. *Let ω be an element of $\Omega^n(\mathbb{R}^n)$ with compact support, and $\int_{\mathbb{R}^n} \omega = 0$, we'll prove that there is α with compact support such that $\omega = d\alpha$.*

1. *Prove that*

$$\begin{aligned} H^n(\mathbb{S}^n) &\to \mathbb{R} \\ \omega &\mapsto \int_{\mathbb{S}^n} \omega \end{aligned}$$

is a bijection.

Since $\int_{\mathbb{S}^n} \mathrm{vol}_{\mathbb{S}^n} \neq 0$*, we say the map is not* 0*, hence bijective, then we have the following statement.*

Let $\omega \in \Omega^n(\mathbb{S}^n)$*, assume that* $\int_{\mathbb{S}^n} \omega = 0$ *then* $\omega = d\beta$*.*

2. *Let* $x_0 \in \mathbb{S}^n$*, then* $\mathbb{S}^n \setminus \{x_0\}$ *is diffeomorphism to* $\mathbb{R}^n$*. Just stereographic* ψ*.*

Let $\omega \in \Omega^n(\mathbb{R}^n)$ *with compact support and* $\int_{\mathbb{R}^n} \omega = 0$*.*

$\psi^*\omega$ *is defined on* $\mathbb{S}^n \setminus \{x_0\}$ *and is* 0 *on a neighborhood of* x_0*.*

Let $\omega_0 = 0$ *on a neighborhood of* x_0 *and* $\omega_0 = \psi^*\omega$*,*

$$\int_{\mathbb{S}^n} \omega_0 = \int_{\mathbb{S}^n \setminus \{x_0\}} \psi^*\omega = \int_{\mathbb{R}^n} \omega = 0.$$

Then there is an $(n-1)$*-form* β *on* $\mathbb{S}^n$ *such that* $d\beta = \omega_0$*.*

Note that $d\beta = 0$ *around* x_0*, then by Poincaré lemma, there is* γ*such that* $\beta = d\gamma$ *around* x_0*. We use a cut-off function to extend* γ *to a global form* $\tilde{\gamma}$ *on* $\mathbb{S}^n$*. Then define* $\alpha_0 = \beta - d\tilde{\gamma}$*, then* α_0 *is* 0 *around* x_0 *and* $d\alpha_0 = d\beta = \omega_0$*. Hence* $\alpha = (\psi^{-1})^*\alpha$ *is the* α *with* $d\alpha = \omega$*.*

Exercise 9.3. *If* $\omega \in \Omega^n(M)$ *with* $\dim M = n \geq 2$*.* M *is connected, oriented and closed. If* $\int_M \omega = 0$*, then* ω *is exact.*

Proof. $M = \bigcup_{i=1}^{N} U_i$ where U_i are diffeomorphism to balls. Let ψ_i be a partition of unity subordinated to U_i, $\sum \psi_i = 1$, $\psi_i \geq 0$ and $\mathrm{Supp}\, \psi_i \subset U_i$.

Let $m_i \in U_i$ and let $\mathscr{O}_i$ open set such that $m_i \in \mathscr{O}_i \subset U_i$. We want to show that $\forall \mathscr{O}_i$, there is a from $\omega_i \in \Omega^n(M)$ with $\mathrm{Supp}\, \omega_i \subset \mathscr{O}_i$ and $[\omega] = [\sum \omega_i]$.

First we try $\tilde{\omega}_i = \psi_i \omega$, then $\omega = \sum \tilde{\omega}_i$. But $\mathrm{Supp}\, \tilde{\omega}_i$ is not necessarily contained in $\mathscr{O}_i$. Hence we want to find α_i with $\mathrm{Supp}\, \alpha_i \subset \mathscr{O}_i$ and $[\alpha_i] = [\tilde{\omega}_i]$.

There exists β_i with $\mathrm{Supp}(\beta_i) \subset \mathscr{O}_i$ and $\int_M \beta_i \neq 0$. We define $\alpha_i = \frac{\int \tilde{\omega}_i}{\int \beta_i} \beta_i$. Then $\int \alpha_i = \int \tilde{\omega}_i$ on U_i, hence $\alpha_i - \tilde{\omega}_i = d\gamma_i$ with support in U_i.

Let $\tilde{\gamma}_i = \gamma_i$ on U_i, 0 outside. $\omega_i = \alpha_i$ on U_i and 0 outside. Then $\omega_i - \tilde{\omega}_i = d\gamma_i$. Then $[\omega] = \sum [\omega_i]$.

Now we want to move these ω_i into one chart!

Let U be an open set in M which is diffeomorphisms to $\mathbb{R}^n$. Let $q_1, \cdots, q_p$ in U distinct points. Then by the Theorem 13.3, there exists a diffeomorphism F such that $F(q_i) = m_i$. Choose $\mathscr{O}_i$ such that $F^{-1}(\mathscr{O}_i) \subset U$. Then we have

$$[F^*\omega] = \sum [F^*\omega_i],$$

by our construction $\mathrm{Supp}(\sum F^*\omega_i) \subset U$.

Let $\beta = \sum F^*\omega_i$ then $\mathrm{Supp}(\beta) \subset U$, then

$$\int_M \beta = \sum_M \int F^*\omega_i = \sum_M \int \omega_i = 0.$$

Thus $\beta = d\alpha$ with $\mathrm{Supp}(\alpha) \subset U$ hence β is closed. $\square$

Now we conclude what we've proved: if M is a closed, oriented and connected, then $\int_M \omega = 0 \iff \omega = d\alpha$, which indicates $b^n(M) = 1$.

We will see that if M is closed but not oriented, then $H^n(M) = 0$.

10 Vector Fields and Flows

10.1 Differential equations

Definition 10.1. *A vector field on $\mathbb{R}^n$, defined on $\mathscr{O} \subset \mathbb{R}^n$ is $X : \mathscr{O} \to \mathbb{R}^n$.*

The associated differential equation is

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n).$$

An orbit of the differential equation is a solution $c(t) = (x_1(t), \cdots, x_n(t))$ of the last equation, i.e. $\frac{dc}{dt}(t) = X(c(t))$.

Definition 10.2. *A smooth vector filed ξ is a map $M \xrightarrow{\xi} TM = \bigsqcup_x T_x M$ such that*

(i) $\xi(m) \in T_m M$;

(ii) for every m, there exists a chart $\varphi = (x_1, \cdots, x_n)$ locally $\xi = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial x_i}$, where ξ_i are smooth function.

Definition 10.3. *An **orbit** of ξ is a curve $c :]a, b[\to M$, such that $\dot{c}(t) = \xi(c(t))$.*

Definition 10.4. *A **flow** of a vector field ξ on M is a map $\phi : \mathscr{O} \to M$, where $\mathscr{O}$ is an open subset in $M \times \mathbb{R}$ containing $M \times \{0\}$. We use I_m to denote $\mathscr{O} \cap \{m\} \times \mathbb{R}$, ϕ satisfies*

(i) $\phi(m, 0) = m$.

(ii) the map $\phi|_{I_m} : (m, t) \mapsto \phi_t(m) := \phi(m, t)$ is an orbit of the vector field ξ, i.e.

$$\frac{d}{dt}\Big|_{t=s} \phi_t(m) = \xi(\phi_s(m)).$$

Definition 10.5. *We say ϕ is **maximal**, if for any flow $(\phi', \mathscr{O}')$ then $\mathscr{O}' \subset \mathscr{O}$ and $\phi|_{\mathscr{O}'} = \phi'$.*

Theorem 10.1. *Let ξ be a smooth vector field on M, then ξ admits a unique maximal flow ϕ.*

Remark 10.1. *In $\mathbb{R}^n$: existence and uniqueness of solution of ODE.*

Definition 10.6. *A flow is **complete** if $\mathscr{O} = M \times \mathbb{R}$.*

Theorem 10.2. *If ξ has compact support; then its maximal flow is complete.*

Definition 10.7. *A vector field is **complete** whenever its maximal flow is complete.*

Remark 10.2. ***Convention.*** *Assume that for simplicity the flow ϕ, $\{\phi_t\}_{t \in \mathbb{R}}$ is complete. Then*

$\phi_t \circ \phi_s = \phi_{t+s}$: $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$.

$$\left(\frac{d}{dt}\phi_t\right)\Big|_{t=u} (\phi_s(x)) = \xi(\phi_u(\phi_s(x))).$$

$$\frac{d}{dt}\Big|_{t=u} (\phi_{t+s}(x)) = \frac{d}{dw}\Big|_{w=u+s} \phi_w(x) = \xi(\phi_{s+u}(x)).$$

Let $c_1 : t \mapsto \phi_t(\phi_s(x))$ thus it is a solution of

$$\begin{cases} \dot{c_1}(t) = \xi(c_1(t)), \\ c_1(0) = \phi_s(x). \end{cases}$$

Let $c_2 : t \mapsto \phi_{t+s}(x)$ and it also a solution of the last equations. By the uniqueness, $c_1(t) = c_2(t)$ for any $t \in \mathbb{R}$.

Definition 10.8. ***A vector field depending on time*** *is a family of vector field $\{\xi_t\}_{t\in\mathbb{R}}$ such that (locally) $\xi_t = \sum_{i=1}^{n} f_i(t, x_1, \cdots, x_n)\frac{\partial}{\partial x_i}$, where the f_i are smooth.*

The integral curve of *ξ_t is a differentiable curve $\gamma : J_0 \to M$, where J_0 is an interval contained in the domain of t, such that*

$$\gamma'(t) = \xi_t(\gamma(t)).$$

The flow of a vector field depending on time *is $\phi : \mathscr{O} \to M$, where $\mathscr{O}$ is an (good) open subset in $M \times \mathbb{R} \times \mathbb{R}$ containing $M \times \{(s,s) \in \mathbb{R}^2\}$, denoting $\phi_s^u(x) = \phi(x, s, u)$, we ask*

$$\begin{cases} \phi_s^s(x) = x, \\ \frac{\partial}{\partial s}\phi_s^u(x)\Big|_{s=t} = \xi_t(\phi_t^u(x)). \end{cases}$$

In other words, $c : t \mapsto \phi_t^u(x)$,

(i) is a solution of

$$\dot{c}(t) = \xi_t(c_t(x)),$$

(ii) $c(u) = x$.

Especially we have $\frac{\partial}{\partial s}\phi_s^u(x)\big|_{s=u} = \xi_u(\phi_u^u(x)) = \xi_u(x)$.

Exercise 10.1. *$\phi_v^u \circ \phi_u^s = \phi_v^s$. (Similar to prove $\phi_t \circ \phi_s = \phi_{t+s}$.)*

Remark 10.3. *A (usual) flow is a flow depending on time $\xi_t = \xi$, then $\phi_t^s = \phi_{t-s}$.*

We will prove the existence and uniqueness of flows depending on time.

A vector filed depending on time, is a vector field on $M \times \mathbb{R}$, $\xi(m,t) = \xi_t(m)$.

10.2 Lie brackets

Definition 10.9. *A* ***derivation at a point*** *$m \in M$ is a linear map $\partial : C^\infty(M) \to \mathbb{R}$ such that*

(i) $\partial(fg) = f(m)\partial(g) + g(m)\partial(f)$,

(ii) $\partial f = 0$ if $f \equiv 0$ on $V(m)$.

A ***derivation on*** *M is a linear map $\partial : C^\infty(M) \to C^\infty(M)$.*

(ii) indicates that if $f = g$ on $V(m)$, then $\partial f = \partial g$.

If a function f is just defined on $U \in V(m)$, then we can define uniquely

$$\partial f := \partial(\psi f),$$

where $\operatorname{Supp}\psi \subset U$ and $\psi \equiv 1$ on $V(m)$. (It does not depend on the choice of ψ.)

Theorem 10.3. *(a) Every vector X in T_mM defines a derivation*

$$\partial_X f := df(X).$$

(b) Conversely every derivation is uniquely of this form.

Proof.

(a) Let X be a vector in T_mM, there exists a curve $c(t)$ such that $c(0) = m$ and $\dot{c}(0) = X$, we have

$$df(X) = \frac{d}{dt}\Big|_{t=0}(f \circ c(t)),$$

then (a) follows from derivation of products.

(b) If f is defined on $U \in V(m)$, let X be coordinates on U with $X(m) = 0$. By Taylor's formula,

$$f = f(m) + \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} h_i x_i,$$

where a_i are constant and h_i are smooth function with $h_i(m) = 0$.

Let ∂ be a derivative, then

$$\partial f(m) = 0,$$

$$\partial(a_i x_i) = a_i \partial x_i + x_i(m)\partial a_i = a_i \partial x_i,$$

$$\partial(h_i x_i) = h_i(m)\partial x_i + x_i(m)\partial h_i = 0.$$

Hence $\partial f = \sum\limits_{i=1}^{n} a_i \partial x_i$, where $a_i = \frac{\partial f}{\partial x_i}(m)$ and ∂x_i are constant. Then we define the vector field $Y = \sum\limits_{i=1}^{n} (\partial x_i)\frac{\partial}{\partial x_i}$, then

$$df(Y) = \sum_{i=1}^{n} a_i dx_i(\sum_{j=1}^{n}(\partial x_j)\frac{\partial}{\partial x_j}) = \sum_{i=1}^{n} a_i \partial x_i = \partial f.$$

□

Definition 10.10. *A* ***derivation on a manifold*** *is a linear map $\partial : C^\infty(M) \to C^\infty(M)$ with two properties*

(i) If $f \equiv 0$ on $V(m)$, then $\partial f \equiv 0$ on $V(m)$.

(ii) $\partial(fg) = f\partial g + g\partial f$.

Theorem 10.4. *(a) Every vector filed X on M defines a derivative on M by $\partial_X f := df(X)$.*

(b) Every derivative on M is obtained by a unique vector field.

Proposition 10.1. *If ∂_1 and ∂_2 are two derivations, then*

$$[\partial_1, \partial_2] : f \mapsto \partial_1(\partial_2(f)) - \partial_2(\partial_1(f))$$

is also a derivation.

Then we have the Jacobi identity,

$$\big[\partial_1, [\partial_2, \partial_3]\big] + \big[\partial_2, [\partial_3, \partial_1]\big] + \big[\partial_3, [\partial_1, \partial_2]\big] = 0.$$

Definition 10.11. *Given two vectors fields X, Y, the Lie bracket $[X, Y]$ is the vector field such that*

$$\partial_{[X,Y]} = [\partial_X, \partial_Y].$$

Remark 10.4. *The following notations stand for the same thing:*

$$df(X), \quad \partial_X f, \quad L_X f, \quad X \cdot f.$$

For example,

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f),$$

$$L_{[X,Y]} f = L_X(L_Y f) - L_Y(L_X f).$$

Proposition 10.2.

(i) $[X, Y] = -[Y, X]$,

(ii) $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$

(iii) $[fX, Y] = f[X, Y] - (Y \cdot f)X.$

Proof.

$$\begin{aligned} [fX, Y] \cdot g &= (fX) \cdot (Y \cdot g) - Y \cdot (fX \cdot g) \\ &= fX \cdot (Y \cdot g) - (Y \cdot f)(X \cdot g) - fY \cdot (X \cdot g) \\ &= \big(f[X, Y] - (Y \cdot f)X\big) g. \end{aligned}$$

□

Exercise 10.2. $X = \sum\limits_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$*, and* $Y = \sum\limits_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$*, then*

$$[X, Y] = \sum_{i=1}^{n} \left(f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Proof.

$$\begin{aligned} [X, Y] &= \sum_{i,j} [f_i \frac{\partial}{\partial x_i}, g_j \frac{\partial}{\partial x_j}] \\ &= \sum_{i,j} f_i [\frac{\partial}{\partial x_i}, g_j \frac{\partial}{\partial x_j}] - g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i} \\ &= \sum_{i,j} f_i (g_j [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] + \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j}) - g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i} \\ &= \sum_{i,j} f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i} \\ &= \sum_{i=1}^{n} \left(f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right) \frac{\partial}{\partial x_i} \end{aligned}$$

□

Exercise 10.3. $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}] = \frac{\partial}{\partial z}.$

Definition 10.12. *We define* $L_X Y := [X, Y]$.

Then $[X, fY] = (X \cdot f)Y + f[X, Y]$ can be written into

$$L_X(fY) = (L_X f)Y + f L_X Y.$$

10.3 Linear Differential Fields

Let $A \in M_n(\mathbb{R}^n)$, define $X_A(x) = A \cdot x$, such a vector field is called linear.

Exercise 10.4. $[X_A, X_B] = X_{-[A,B]}$, *where* $[A, B] = AB - BA$.

Proof. Set $A = \{a_i^j\}$ and $B = \{b_i^j\}$. Then $X_A = \sum\limits_{i,j} a_i^j x_j \frac{\partial}{\partial x_i}$, and $X_B = \sum\limits_{i,j} b_i^j x_j \frac{\partial}{\partial x_i}$. $[A, B]_i^j = \sum\limits_k a_i^k b_k^j - b_i^k a_k^j$. $X_{[A,B]} = \sum\limits_{i,j,k} (a_i^k b_k^j - b_i^k a_k^j) x_j \frac{\partial}{\partial x_i}$.

$$[X_A, X_B] = [\sum_{i,j} a_i^j x_j \frac{\partial}{\partial x_i}, \sum_{k,l} b_k^l x_l \frac{\partial}{\partial x_k}] = \sum_{i,j,k,l} a_i^j x_j b_k^l \delta_l^i \frac{\partial}{\partial x_k} - b_k^l x_l \delta_j^k a_i^j \frac{\partial}{\partial x_i} = \sum_{i,j,k} (a_k^j b_i^k - b_k^j a_i^k) x_j \frac{\partial}{\partial x_i}.$$

□

Let X_A be a linear vector field associated to A, what is the flow of X_A is

$$\phi_t(x) = e^{tA} \cdot x.$$

Exercise 10.5. *Let ψ be a diffeomorphism from M to N. Let X be a vector filed on M with flow ϕ_t. Let $\tilde{\phi}_t = \psi \circ \phi_t \circ \psi^{-1}$. Show that $\tilde{\phi}_t$ is the flow of some vector field $Y(y) = (T_{\psi^{-1}(y)}\psi) X(\psi^{-1}(y))$.*

Proof. Set $\phi : M \times \mathbb{R} \to M$, then $\tilde{\phi} : N \times \mathbb{R} N$. For any $n \in N$, set $m = \psi^{-1}(n)$.

(i) $\tilde{\phi}(n, 0) = \psi \circ \phi_0 \circ \psi^{-1}(n) = \psi \circ \phi_0(m) = \psi(m) = n$.

(ii)

$$\frac{d}{dt}\Big|_{t=0} \tilde{\phi}(n, t) = \frac{d}{dt}\Big|_{t=0} (\psi \circ \phi_t \circ \psi^{-1}(n)) = \frac{d}{dt}\Big|_{t=0} (\psi \circ \phi_t(m)) = (T_m \psi) X(m) = (T_{\psi^{-1}(y)}\psi) X(\psi^{-1}(y)).$$

□

Proposition 10.3. *Let X, Y be vector field on M, X has flow ϕ_t and let $Y_t = \phi_{t,*}(Y)$, then*

$$\frac{d}{dt}\Big|_{t=0} Y_t = -[X, Y].$$

Remark 10.5. *Convention:* $\psi^* = (\psi^{-1})_*$.

Hence by defining $Z_t = \phi_t^*(Y)$, we have

$$\frac{d}{dt}\Big|_{t=0} Z_t = [X, Y].$$

Note that $\phi_{-t} : \phi_t(U) \to U$ will push $Y_{\phi_t(u)}$ into $T_u M$.

Let $f : M \to \mathbb{R}$, then

$$L_X f = df(X) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \phi_t) = \frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^*(f)).$$

Labourie's Proof. Let $Z = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\phi_s^*(Y))$. Let $f : M \to \mathbb{R}$.

$$\begin{aligned}
(L_Z f)_m &= (df)_m \left(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} (\phi_s^*(Y))_m \right) \\
&= (df)_m \left(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} (T_{\phi_s(m)}\phi_{-s})(Y_{\phi_s(m)}) \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \left((df)_m (T_{\phi_s(m)}\phi_{-s})(Y_{\phi_s(m)}) \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \left(d_{\phi_s(m)}(f \circ \phi_{-s})(Y_{\phi_s(m)}) \right),
\end{aligned}$$

where the third equality holds since $(df)_m$ is just a linear transformation on T_mM, independent of s, and the fourth equality is the derivative of component function.

Let define $g_s : M \to \mathbb{R}$. $g_s(m) = [d_m(f \circ \phi_{-s})](Y(m)) = L_Y(f \circ \phi_{-s})(m)$. Then

$$(L_Z f)_m = \frac{\mathrm{d}}{\mathrm{d}u}\Big|_{u=0} g_u \circ \phi_u(m) = \frac{\partial}{\partial s}\Big|_{s=0,t=0} (g_s \circ \phi_t)(m) + \frac{\partial}{\partial t}\Big|_{s=0,t=0} (g_s \circ \phi_t)(m).$$

Note that

$$\frac{\partial}{\partial t}\Big|_{t=0,s=0} (g_s \circ \phi_t)(m) = L_X(g_s(m))\Big|_{s=0} = L_X(df(Y)) = L_X L_Y f.$$

$$\frac{\partial}{\partial s}\Big|_{t=0,s=0} (g_s \circ \phi_t)(m) = \frac{\partial}{\partial s}\Big|_{s=0} L_Y \left(f \circ \phi_{-s})(m) = L_Y(\frac{\partial}{\partial s}\Big|_{s=0} f \circ \phi_{-s} \right)(m) = -L_Y L_X f(m).$$

□

My Proof. We prove directly that

$$\frac{d}{dt}\Big|_{t=0} \phi_{-t,*}(Y) = [X, Y].$$

We denote the Jacobi matrix of ϕ_t as A_t, and the Jacobi matrix of ϕ_{-t} as A_t^{-1}, note that $A_t^{-1}(\phi_t(u))A_t(u) = A_t(u)A_t^{-1}(\phi_t(u)) = \mathrm{id} : T_uM \to T_uM$. Note that $A_t(u) = \frac{\partial \phi_t^i}{\partial x^j}(u)$. Derivate $A_t^{-1}(\phi_t(u))A_t(u) = \mathrm{id}$ by t, we have

$$\frac{\partial}{\partial t} A_t^{-1}(\phi_t(u) = -A_t^{-1}(\phi_t(u)) \left(\frac{d}{dt} A_t(u) \right) A_t^{-1}(\phi_t(u)).$$

Let $X = a^i \partial_i$, $Y = b^i \partial_i$, and we use $\{a^i\}$, $\{b^i\}$ to denote these volume matrix, which is a matrix function around u.

$$\phi_{-t,*}(Y) = A_t^{-1}(\phi_t(u))\{b^i(\phi_t(u))\}.$$

Derivate by t, in matrix form, we have

$$\begin{aligned}
L_XY = \frac{d}{dt}\Big|_{t=0}\phi_{-t,*}(Y) &= \left(\frac{d}{dt}\Big|_{t=0}A_t^{-1}(\phi_t(u))\right)\{b^i(\phi_0(u))\} + A_0^{-1}(\phi_0(u))\left(\frac{d}{dt}\Big|_{t=0}\{b^i(\phi_t(u))\}\right)\\
&= -\left(A_t^{-1}(\phi_t(u))\left(\frac{d}{dt}A_t(u)\right)A_t^{-1}(\phi_t(u))\right)\Big|_{t=0}\{b^i(u)\} + \frac{d}{dt}\Big|_{t=0}\{b^i(\phi_t(u))\}\\
&= -\left(\frac{d}{dt}\Big|_{t=0}A_t(u)\right)\{b^i(u)\} + \frac{d}{dt}\Big|_{t=0}\{b^i(\phi_t(u))\}\\
&= -\left(\frac{d}{dt}\Big|_{t=0}\{\frac{\partial\phi_t^i}{\partial x^j}(u)\}\right)\{b^i(u)\} + \{\frac{\partial b^i}{\partial x^j}(u)\}\{\frac{d\phi_t^j(u)}{dt}\Big|_{t=0}\}\\
&= -\left(\frac{\partial\frac{d}{dt}\big|_{t=0}\{\phi_t^i\}}{\partial x^j}(u)\}\right)\{b^i(u)\} + \{\frac{\partial b^i}{\partial x^j}(u)\}\{a^j(u)\}\\
&= -\{\frac{\partial a^i}{\partial x^j}(u)\}\{b^i(u)\} + \{\frac{\partial b^i}{\partial x^j}(u)\}\{a^j(u)\}\\
&= -\frac{\partial a^i}{\partial x^j}(u)b^j(u)\partial_i + \frac{\partial b^i}{\partial x^j}(u)a^j(u)\partial_i\\
&= -YX + XY = [X,Y].
\end{aligned}$$

□

Proposition 10.4. $L_X(fY) = (L_Xf)Y + fL_XY$.

Example 10.1. *When $X = X_A$ and $Y = X_B$,*

$$Y_t(u) = \phi_{t,*}Y(u) = (T_{\phi_{-t}(u)}\phi_t)(Y(\phi_{-t}(u)) = e^{tA}Be^{-tA}u.$$

$$\frac{d}{dt}\Big|_{t=0}Y_t(u) = (AB - BA)u = [X_A, X_B]u = -X_{[A,B]}u.$$

Example 10.2. *Let $\{\phi_t\}$ be the flow of X, and $\{\psi_t\}$ the flow of Y. Assume that $\phi_t\circ\psi_s = \psi_s\circ\phi_t$ for any s,t, then $[X,Y] = 0$.*

What is the flow of Y_s?

$$\tilde{\psi}_t = \phi_s\circ\psi_t\circ\phi_{-s} = \psi_t.$$

In this case, the flow of Y_s is the flow of Y, hence $Y_s = Y$. Then $\frac{d}{ds}Y_s = 0$.

Example 10.3. *Assume $[X,Y] = 0$, then $\phi_t\circ\psi_s = \psi_s\circ\phi_t$ for any s,t.*

$$0 = \frac{d}{ds}[\phi_{t,*}(Y_s)](y) = \frac{d}{ds}(T_{\phi_{-t}(y)}\phi_t)(Y_s(\phi_{-t}(y)) = \left(T_{\phi_{-t}(y)}\phi_t\right)\left(\frac{d}{ds}Y_s(\phi_{-t}(y)\right),$$

hence $\frac{d}{ds}\big|_{s=t}Y_s = 0$, hence $Y_s = Y$ for any s. Then their flows coincide

$$\phi_s\circ\psi_t\circ\phi_{-s} = \psi_t,$$

that is what we need to prove.

Geometric interpretation.

Assume that $[X,Y] = 0$, then we have $\phi_s\circ\psi_t = \psi_t\circ\phi_s$. Given $m_0\in M$, it allows us to define $F:\mathbb{R}^2\to M$, such that

$$F(s,t) := (\phi_s\circ\psi_t)(m_0).$$

Then we have $\frac{\partial F}{\partial s}\ (s_0,t_0) = X(F(s_0,t_0))$ and $\frac{\partial F}{\partial t}\ (s_0,t_0) = Y(F(s_0,t_0))$.

Definition 10.13. *Let ω be a k-form on M, we define the Lie derivative of forms as*

$$L_X\omega := \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\phi_s^*\omega).$$

This is coherent when $\deg\omega = 0$.

Proposition 10.5.

(a). $L_X : \Omega^k(M) \to \Omega^k(M)$ *is linear;*

(b). $L_X(\alpha\wedge\beta) = L_X\alpha\wedge\beta + \alpha\wedge L_X\beta$*;*

(c). $L_X(d\omega) = d(L_X\omega)$.

(d). $L_{[X,Y]}\omega = L_XL_Y\omega - L_YL_X\omega$*;*

(e). ***Lie-Cartan formula***

$$L_X\omega = d(i_X\omega) + i_Xd\omega.$$

Proof. (a) is trivial. (b) is from the fact $\phi_t^*(\alpha\wedge\beta) = \phi_t^*\alpha\wedge\phi_t^*\beta$. (c) is shown in the following calculation,

$$L_X(d\omega) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\phi_s^*d\omega) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(d\phi_s^*\omega) = d\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\phi_s^*\omega) = dL_X\omega.$$

The proof of (d) is by induction on the degree of ω. For $\deg\omega = 0$, it is the definition of $[X,Y]$. Assume that it is true for $\deg\eta = p-1$,

$$L_{[X,Y]}\eta = L_XL_Y - L_YL_X\eta.$$

Any form of degree p can be written into the following form

$$\omega = \sum_{i\in I} f_i d\alpha_i,$$

where α_i are $p-1$ forms. Now

$$\begin{aligned}
L_{[X,Y]}\omega &= L_{[X,Y]}\left(\sum_{i\in I} f_i d\alpha_i\right)\\
&= \sum_{i\in I}(L_{[X,Y]}f_i)d\alpha_i + f_i(L_{[X,Y]}d\alpha_i)\\
&= \sum_{i\in I}(L_{[X,Y]}f_i)d\alpha_i + f_id(L_{[X,Y]}\alpha_i)\\
&= \sum_{i\in I}\big((L_XL_Y - L_YL_X)f_i\big)d\alpha_i + f_id(L_XL_Y - L_YL_X)\alpha_i\\
&= \sum_{i\in I}\big((L_XL_Y - L_YL_X)f_i\big)d\alpha_i + f_i(L_XL_Y - L_YL_X)d\alpha_i\\
&= \sum_{i\in I}(L_XL_Y - L_YL_X)(f_id\alpha_i)\\
&= (L_XL_Y - L_YL_X)\left(\sum_{i\in I} f_id\alpha_i\right)\\
&= (L_XL_Y - L_YL_X)\omega.
\end{aligned}$$

□

Exercise 10.6. *Any form of degree p can be written into the following form*

$$\omega = \sum_{i \in I} f_i d\alpha_i,$$

where α_i are $p-1$ forms.

The First Proof of Lie-Cartan formula. For $\deg \omega = 0$, set $\omega = f$, then

$$i_X df + di_X f = i_X df = df(X) = L_X f.$$

Or we can also prove for $\deg \omega = 1$.

Now we assume the formula is true for $\deg(p-1)$.

First case is $\omega = f d\alpha$, where $\deg \alpha = p-1$, then

$$\begin{aligned} L_X \omega &= L_X(f d\alpha) \\ &= (L_X f) d\alpha + f d L_X \alpha \\ &= (L_X f) d\alpha + f d(i_X d\alpha + d i_X \alpha) \\ &= (L_X f) d\alpha + f d i_X d\alpha. \end{aligned}$$

$$\begin{aligned} i_X d\omega + di_X \omega &= i_X d(f d\alpha) + d i_X (f d\alpha) \\ &= i_X (df \wedge d\alpha) + i_X (f dd\alpha) + d(f i_X d\alpha) \\ &= (i_X df) d\alpha - df \wedge i_X(d\alpha) + df \wedge i_X(d\alpha) + f d i_X d\alpha \\ &= (L_X f) d\alpha + f d i_X d\alpha \end{aligned}$$

Then this formula is true for $\omega = f d\alpha$. We deduce Lie-Cartan is true for all ω of degree p, using the linearity of L_X, $d \circ i_X$ and $i_X \circ d$. □

Now we check the Special case of Lie-Cartan formula 7.1.

$J_s(m) = (m, s)$ and $\phi_s : (m,t) \mapsto (m, t+s)$ is the flow of ∂_t. $J_s = \phi_s \circ J_0$ and $(J_s)^* = J_0^* \phi_s^*$.

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} (\phi_s^* \alpha) = L_{\partial_s} \alpha = i_{\partial_s} d\alpha + d i_{\partial_s} \alpha$$

We pull them back through J_0^*, getting

$$J_0^* \left(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} (\phi_s^* \alpha) \right) = \left(\frac{\mathrm{d}}{\mathrm{d}s} (J_0^* \circ \phi^*)\Big|_{s=0} \right) (\alpha) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} (J_s^* \alpha).$$

10.4 Frobenius Theorem

Definition 10.14. *M is a manifold, $TM = \sqcup_{x \in M} T_x M$. A **sub-distribution** (or a distribution) of rank p, is a family $\{\mathcal{P}_x\}_{x \in M}$ such that for any x, P_x is a vector subspace of dimension p pf $T_x M$.*

*A distribution $\mathcal{F}$ of rank p is **smooth** if for every m, there exists smooth vector fields $X_1, \cdots, X_p$ on a neighborhood of x, such that $X_1(n), \cdots, X_p(n)$ is a basis of $\mathcal{F}_n$.*

For example, (i) a vector field X such that $X(m) \neq 0$ on M, then $\mathcal{P}_m = \mathbb{R}X(m)$.

(ii) Let U be an open set in $\mathbb{R}^n \times \mathbb{R}^k$, $\mathcal{P}_{(m,n)} = \{0\} \times \mathbb{R}^k \subset T_{(m,n)}\mathbb{R}^n \times \mathbb{R}^k$.

(iii) If U is an open set in $M \times N$. $\mathcal{P}_{(m,n)} = T_n N \times \{0\} \subset T_{(m,n)} M \times N$.

Definition 10.15. *A distribution is called* ***integrable*** *if for any $x \in M$, there is a chart (U, X) at x such that $X_*(\mathcal{F})$ is of type (ii).*

Or equivalently, a distribution is called ***integrable*** *if for any $m \in M$, there is a submanifold $N_m \ni m$, such that $\forall x \in N$, $T_x N_m = \mathcal{F}_x$.*

Exercise 10.7. *If we can find $X_1, \cdots, X_p$ as above that that $[X_i, X_j] = 0$, then $\mathcal{F}$ is integrable.*

Proposition 10.6 (Pre Frobenius)**.** *Assume that on a neighborhood of m (any $m \in M$), there exist k-vector fields defined on U,*

(i) $X_1(n), \cdots, X_k(n)$ is a basis of $\mathcal{F}_n$, $\forall n \in U$,

(ii) $[X_i, X_j] = 0$ for any i, j.

Then $\mathcal{F}$ is integrable.

Proof. Let $m \in M$, let ϕ_t^i is the flow of X_i. We know that $\phi_t^i \circ \phi_s^j = \phi_s^j \circ \phi_t^i$ (condition (ii)).

Define

$$\begin{aligned} &\psi :]-\varepsilon, \varepsilon[^k \to M \\ &(t_1, \cdots, t_k) \mapsto (\phi_{t_1}^1 \circ \cdots \circ \phi_{t_k}^k)(m). \end{aligned}$$

Then

$$T_{(t_1, \cdots, t_k)}\psi \left(\frac{\partial}{\partial t_1} \right) = \frac{\partial}{\partial t_1}\Big|_{s=t_1} \phi_s^1 \circ \phi_{t_2}^2 \circ \cdots \circ \phi_{t_k}^k(m) = X_1(\phi_{t_1}^1 \circ \cdots \circ \phi_{t_k}^k(m)).$$

Similarly, due to ϕ_s^i and ϕ_t^j commutes, $T\psi(\frac{\partial}{\partial t_j}) = X_j(\phi_{t_1}^1 \circ \cdots \circ \phi_{t_k}^k(m))$. Then

$$T_{\psi(T)}\big(\psi(]-\varepsilon, \varepsilon[^k)\big) = \operatorname{Im}(T_T\psi) = \operatorname{Span}(X_1, \cdots, X_k) = \mathcal{F}_{\psi(T)},$$

where T standing for $(t_1, \cdots, t_k)$. $\square$

Theorem 10.5 (Frobenius Theorem)**.** *$\mathcal{F}$ is integrable, if an only if $\forall X, Y$ such that $X(m) \in \mathcal{F}_m$ and $Y(m) \in \mathcal{F}_m$ then $[X, Y](m) \in \mathcal{F}_m$.*

Remark 10.6. *If X is a vector field, for any m, $X(m) \neq 0$. Define $\mathcal{L}_x = \mathbb{R}X \subset TM$, then the 1-dimension distribution is integrable ($\iff$ existence of solution ODE).*

Hence every 1-dimension distribution is integrable.

Non-Example. On $\mathbb{R}^3$, $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. Then $\mathcal{F}_{(x,y,z)} = \langle X, Y \rangle$ is not integrable. Note that $(X, Y, [X, Y])$ is always a basis of $\mathbb{R}^3$.

Proof of Frobenius Theorem. $\Rightarrow$ is note difficult due to the definition of integral.

For $\Leftarrow$, we will prove that there exists $X_1, \cdots, X_k$ a basis of $\mathscr{F}$, with $[X_i, X_j] = 0$. $\square$

11 Vector Bundle

11.1 Definitions

$TM = \sqcup_x T_x M$, $\mathcal{F} = \sqcup_x \mathcal{F}_x$, $\wedge^k(T^*M) = \sqcup_x T_x^* M$ denoting family of vector spaces.

Definition 11.1. *A **vector bundle of rank** k is a triple $(\pi, \mathcal{E}, X)$ where $\mathcal{E}, X$ are topological spaces (which are nice: Hausdorff and σ-compact), and $\pi : \mathcal{E} \to X$ is continuous.*

*We call π as **projection**, $\mathcal{E}$ the **total space** and X is the **base space**.*

*(i) The **fiber** at x, $\mathcal{E}_x = \pi^{-1}(x)$ is a vector space of dimension k.*

*(ii) **Local trivialization property**: given $x \in X$, there exists a neighborhood U of x (**trivializing neighborhood**), and a continuous map ϕ (called **trivializing**),*

$$\pi^{-1}(U) =: \mathcal{E}|_U \xrightarrow{\phi} E \times U,$$

where E is a vector space, such that

(i) $\phi(\mathcal{E}_x) = E \times \{x\}$,

(ii) $\phi|_{\mathcal{E}_x}$ is a linear isomorphism with $E \times \{x\}$.

Example 11.1. *Trivial bundle over X, for E any vector space, $\mathcal{E} = E \times X$ with $\pi(e, x) = x$.*

Example 11.2. ***Tautological bundle.***

$G_k(E) = \{P$ vector space in E of dimension $k\}$.

$$E \times G_k(E) \supset \tau_k = \{(u, P) \in E \times G_k(E) : u \in P\}.$$

$$\pi : \tau_k \to G_k(E), (u, P) \mapsto P.$$

The fiber $\pi^{-1}(P) = \{(u, P) : u \in P\} \approx \{u \in P\}$ a vector space of dimension k.

Let $P \in G_k(E)$ and Q a vector space such that $P \oplus Q = E$. We defined

$$U_{P,Q} = \{P' \in G_k(E) : P' \oplus Q = E\}.$$

For every $P' \in U_{P,Q}$, let $\lambda_{P'} : P' \to P$ such that $x - \lambda_{P'}(x)$ is parallel to Q.

$$\begin{aligned} \phi : \tau_k|_{U_{P,Q}} &\to P \times U \\ (v, P') &\mapsto (\lambda_{P'}(v), P). \end{aligned}$$

Now $\phi|_{\pi^{-1}(P')} = \lambda_{P'}$ and $\lambda_{P'}$ is an isomorphism.

Define $\hat{\phi} : E \times U_{P,Q} \to P \times U_{P,Q}$, $(u, P') \mapsto (\lambda(u), P')$, where λ is the projection from E to P such that $E(x) - x$ parallel to Q. Note that $\lambda_{P'} = \lambda|_{P'}$ and $\phi = \hat{\phi}|_{\tau_k|_{U_{P,Q}}}$.

Exercise 11.1. *Accept the fact that the total space of the tautological bundle of $\mathbb{RP}^1$ is the Möbius band.*

Definition 11.2. *A **continuous section** of $\pi : \mathcal{E} \to X$ is a continuous map $\sigma : X \to \mathcal{E}$ such that $\sigma(x) \in \mathcal{E}_x$.*

For example, Zero section $\sigma_0 : x \to 0_x$ the zero of $\mathcal{E}_x$.

Space of section is denoted by $\Gamma(\mathcal{E})$, it forms a vector space.

Morphism is

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\phi"] \arrow[d] & \mathcal{F} \arrow[dl] \\ X & \end{tikzcd}$$

with $\phi(\mathcal{E}_x) = \mathcal{F}_x$ and $\phi|_{\mathcal{E}_x}$ is linear.

Definition 11.3. *Let $\mathcal{E}$ be a vector bundle, a **sub-bundle** is a closed subset $\mathcal{F} \subset \mathcal{E}$. such that $\mathcal{F} \cap \mathcal{E}_x$ is a vector subspace of $\mathcal{E}_x$.*

Proposition 11.1. *Every sub-bundle is a vector bundle such that the injection is a morphism.*

Example 11.3. *Let $\mathcal{F}$ be a smooth distribution, then $\mathcal{F}$ is a sub-bundle of TM (whether $\mathcal{F}$ integral or not).*

Theorem 11.1. *Every bundle over X (compact) is (isomorphic to) a sub-bundle of the trivial bundle over X.*

Definition 11.4. *$\mathcal{E} \to X$ is a vector bundle, and $\varphi : Y \to X$ continues,*

$$\begin{tikzcd} \varphi^*(\mathcal{E}) \arrow[r, "\Phi"] \arrow[d] & \mathcal{E} \arrow[d] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}.$$

Here

$$\varphi^*(\mathcal{E}) := (u, y) \in \mathcal{E} \times Y : u \in \mathcal{E}_{\varphi(y)}\}.$$

$\pi : \varphi^*(\mathcal{E}) \to Y$, $(u, y) \mapsto y$. $\pi^{-1}(y) \approx \mathcal{E}_{\varphi(y)}$.

Proposition 11.2. *$\varphi^*(\mathcal{E})$ has the structure of a vector bundle, moreover, there is $\Phi : \varphi^*(\mathcal{E}) \to \mathcal{E}$, and $\Phi : (\varphi^*\mathcal{E})_y \to \mathcal{E}_{\varphi(y)}$ is isomorphism.*

*$\varphi^*(\mathcal{E})$ is called the **induced vector bundle** by φ.*

Definition 11.5. *The cocycle point of view.*

Let $\pi : \mathcal{E} \to X$ a vector bundle. Let $\{U_i\}_{i \in I}$ be a covering of X by trivializing neighborhood.

$$\begin{tikzcd} & \mathcal{E}|_{U_i \cap U_j} \arrow[dl, "\phi_j"'] \arrow[dr, "\phi_i"] & \\ (U_i \cap U_j) \times E \arrow[rr, "\psi^{ij}"] & & (U_i \cap U_j) \times E \end{tikzcd}$$

$\psi^{ij}(x, u) = (x, \psi^{ij}(x)(u))$, *where* $\psi^{ij} : U_i \cap U_j \to GL(E)$.

*For any $x \in U_i \cap U_j \cap U_k$, we have the **cocycle condition***

$$\psi_x^{ki} \circ \psi_x^{ij} = \psi_x^{kj}.$$

Given a vector bundle and trivializing covering, we get the cocycle $\psi^{ij} : U_i \cap U_j \to GL(E)$.

Theorem 11.2. *Assume we have a covering $\{U_i\}$ of X and a cocycle $\psi^{ij} : U_i \cap U_j \to GL(E)$, then there is a vector bundle $\mathcal{E} \to X$ whose cocycle is ψ^{ij}.*

Sketch of the proof. Define

$$V = \sqcup_{i\in I}(U_i \times E) = \{(i, x, v) : i \in I, x \in U_i, v \in E\}.$$

We need to glue back fibers over $U_i \cap U_j$.

Let we introduce an equivalence relation on V,

$$(i, x, u) \sim (j, x, v) \iff v = \psi^{ji}(x)u.$$

The cocycle condition tells us this is an equivalence relation. Define $\mathcal{E} = V/\sim$.

We need to show that V is a topological space, and the topology on $\mathcal{E}$ is the quotient topology. $\pi : [(i, x, u)] \mapsto x$ is a projection. □

$\mathcal{E}, \mathcal{F} \to X$ two vector bundles, let $\{U_i\}$ be trivializing covering of $\mathcal{E}$ and $\mathcal{F}$, $\psi^{ij}_{\mathcal{E}} : U_i \cap U_j \to GL(E)$ and $\psi^{ij}_{\mathcal{F}} : U_i \cap U_j \to GL(E)$.

$GL(E \oplus F) \supset GL(E) \times GL(F)$,

$$\tilde{\psi}^{ij} = \begin{pmatrix} \psi^{ij}_{\mathcal{E}} & \\ & \psi^{ij}_{\mathcal{F}} \end{pmatrix},$$

satisfies the cocycle condition. The associated vector bundle is $\mathcal{E} \oplus \mathcal{F}$ and the fiber $(\mathcal{E} \oplus \mathcal{F})_x = \mathcal{E} \oplus \mathcal{F}_x$.

Exercise 11.2. *$\mathcal{E} \otimes \mathcal{F} = \mathcal{E}_x \otimes \mathcal{F}_x$, prove it with cocycles.*

Let M be a manifold with atlas (U_i, X_i), $X_i : U_i \to \mathscr{O} \subset \mathbb{R}^n$.

$$\begin{aligned} U_i \cap U_j &\to GL(\mathbb{R}^n) \\ x &\mapsto \varphi^{ji}_{(x)} = (T_x X^j)(T_x X^i)^{-1} \in GL(\mathbb{R}^n). \end{aligned}$$

An atlas gives $\hookrightarrow$ cocycle $\hookrightarrow$ a vector bundle structure on TM.

$$TM|_{U_i} \xrightarrow{\phi_i} U_i \times \mathbb{R}^n.$$

$$\phi_i(u) = \{x\} \times (a_1, \cdots, a_n), u = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}.$$

Definition 11.6 (Smooth Vector Bundles over Manifolds)**.** *$\pi : \mathcal{E} \to X$ smooth, $\mathcal{E}, X$ are manifolds. We also need the trivializing maps are smooth.*

The cocycle point of view, $\psi^{ij} : U_i \cap U_j \to GL(E)$ are smooth.

TM, $\wedge^k(TM)$, T^*M are smooth bundles. If $\mathcal{E}, \mathcal{F}$, then $\mathcal{E} \oplus \mathcal{F}$ ... are smooth.

11.2 Moser Theorem and Flow-Box

Let M be a compact oriented manifold of dimension n, $\partial M = \varnothing$.

Let $\omega \in \Omega^n(M)$, then

$$\int_M \omega = 0 \iff \omega \text{ is exact.}$$

Theorem 11.3 (Moser)**.** *Let ω_0 and ω_1 be two volume forms on M. Assume that $\int_M \omega_0 = \int_M \omega_1$, then there exists φ preserving the orientation, such that*

$$\varphi^*\omega_0 = \omega_1.$$

Hint: use flows, find a vector field depending on time.

Exercise 11.3. *Let ω be volume form on E a vector space of dimension n.*

$$E \xrightarrow{\Psi} \wedge^{n-1}(E^*), \quad u \mapsto i_u\omega.$$

Prove that Ψ is an isomorphism.

Proposition 11.3. *If ψ_t is 1-parameter of diffeomorphism then*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\alpha)\Big|_{t=0} = L_X\alpha,$$

where $X(m) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\psi(t)(m))$.

Proof of Moser theorem. Since $\int_M \omega_1 - \omega_0 = 0$, then there exists β such that $d\beta = \omega_0 - \omega_1$.

Set $\omega_t = t\omega_1 + (1-t)\omega_0$, $t \in [0,1]$. Note that ω_t is a volume form. By Exercise 12.3, there is X_t a vector filed depending on time such that $i_{X_t}\omega_t = \beta$. Hence

$$\omega_0 - \omega_1 = d\beta = di_{X_t}\omega_t = L_{X_t}\omega_t.$$

Let $\phi_s^u : M \to M$ is the flow with respect to X_t. Note first

$$\phi_u^s \circ \phi_t^u = \phi_t^s.$$

With the proposition 12.1,

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\big((\phi_t^s)^*\omega_t\big) = L_{X_t}\omega_t = \omega_0 - \omega_1,$$

since $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=t}\phi_t^s(m) = X_t(m)$.

We now show that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=t}\big((\phi_t^s)^*\omega_s\big) = 0$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\big((\phi_t^s)^*\omega_s\big) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}(\phi_t^s)^*\omega_t + (\phi_t^t)^*\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\omega_s \\
&= L_{X_t}\omega_t + (\omega_0 - \omega_1) \\
&= (\omega_0 - \omega_1) + (\omega_1 - \omega_0) \\
&= 0.
\end{aligned}$$

Finally, because $(\phi_t^s)^* = (\phi_t^0)^* \circ (\phi_0^s)^*$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\big((\phi_t^s)^*\omega_s\big) = (\phi_t^0)^* \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}\big((\phi_0^s)^*\omega_s\big),$$

hence $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=t}\big((\phi_0^s)^*\omega_s\big) = 0$ since ϕ_t^0 is a diffeomorphism. Now

$$\omega_0 = (\phi_0^0)^*\omega_0 = (\phi_0^1)^*\omega_1.$$

□

Theorem 11.4 (Existence of a flow-box)**.** *Let X be a vector field, let m such that $X(m) \neq 0$, then there is a chart (U, φ) at m such that $\varphi_*(\frac{\partial}{\partial t}) = X$, where $\varphi : U \to \mathscr{O} \subset \mathbb{R} \times E$.*

Proof. 1. Given X and m, there is a submanifold N of dimension $n-1$ ($n = \dim M$), such that for any $x \in N$, $X(x) \notin T_x N$.

2. Produce Ψ a local diffeomorphism from $N\times]-\varepsilon, \varepsilon[\to M$ such that $\Psi_*(\partial_t) = X$.

Let (ϕ_t) be the flow of X, then we define $\Psi(x,t) = \phi_t(x)$. We need to prove that Ψ is a local diffeomorphism.

$T_{(m,0)}\psi = T_{(m,0)}N \oplus \mathbb{R}$. Let $u \in T_{(m,0)}N$, we have $(T_{(m,0)}\Psi)(u,0) = u$. $T_{(m,0)}\Psi(0,1) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\Psi(m,s) = X(m)$. Hence $T_{(m,0)}\Psi$ is invertible, by local immersion theorem, it is a local diffeomorphism. □

Proposition 11.4. *$\mathcal{F}$ is a sub-bundle of $\mathcal{E}$, that is, $\mathcal{F}$ is a closed subset of $\mathcal{E}$ and $\mathcal{F} \cap \mathcal{E}_x$ is a vector space. Then $\mathcal{F}$ is a vector bundle over X.*

Proof. It is enough to prove this property whenever $\mathcal{E} = E \times X$ a trivial vector bundle.

1. Show that $\dim(\mathcal{F} \cap \mathcal{E}_x)$ is constant. ($\mathcal{F}$ is closed subset).

2. You want to find $U \subset X$ such that $\mathcal{F}_U$ can be trivialized.

Let $x_0 \in X$, and $\mathcal{P}_0$ a subspace in E, such that $\mathcal{P}_0 \oplus \mathcal{F}_{x_0} = E$.

Claim: there exists a neighborhood U of x_0 such that for any $x \in U$, $\mathcal{P}_0 \oplus \mathcal{F}_x = E$.

Let g be a euclidean metric on E. Assume there is $(x_i) \to x_0$, by contradiction such that $\mathcal{P}_0 \cap F_{x_i} \neq \{0\}$. Let $u_i \in \mathcal{P}_0 \cap \mathcal{F}_{x_i}$ and $|u_i| = 1$. Extracting a converging subsequence to u_0, then $u_0 \in P_0$ and $u_0 \in \mathcal{F}_{x_0}$ (since $\mathcal{F}$ is closed), then we get a contradiction.

Then there is a trivialization of $\mathcal{F}|_U$,

$$\mathcal{F}|_U \xrightarrow{\psi} \mathcal{F}_{x_0} \times U, \quad v \in \mathcal{F}_x \mapsto \big(\pi_x(v), x\big),$$

where π_x is the projection from $\mathcal{F}_x$ to $\mathcal{F}_{x_0}$ parallel to $\mathcal{P}_0$.

ψ is continuous, being the restriction of a continuous map π to a closed subset. $\pi : \mathcal{E} = E \times U \to \mathcal{F}_{x_0} \times U$. □

12 Connection

From now no, we speak about smooth real vector bundle, over smooth manifolds.

Space of sections of $\mathcal{E} \to M$ is denoted by $\Gamma(\mathcal{E})$. Special notation, $\Gamma(TM) = \chi(M)$, $\Gamma(\wedge^k(T^*M)) = \Omega^k(M)$.

Important construction: $\mathcal{F} = \wedge^k(T^*M) \otimes \mathcal{E} \to M$, the fiber of $\mathcal{F}$ at m is

$$\mathcal{F}_m = \wedge^k(T_m^*M) \otimes \mathcal{E}_m = \{k - \text{forms on } T_M \text{ with values in } \mathcal{E}_m\}.$$

Denoted by $\Omega^k(M;\mathcal{E}) := \Gamma(\wedge^k(T^*M) \otimes \mathcal{E})$.

Goal: for any two points $x, y \in M$, using an extra structure (Connection) and a curve c from x to y, getting a linear isometry from $\mathcal{E}_x$ to $\mathcal{E}_y$ (Parallel Transport).

12.1 Connection and Parallel Transport

Definition 12.1. *a (Koszul-) connection $\mathcal{E} \to M$ is a linear map $\Gamma(TM) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, $(X, \sigma) \mapsto \nabla_X \sigma$, satisfying for any $f \in C^\infty(M)$,*

(i) $\nabla_{fX} s = f \nabla_X s$.

(ii) $\nabla_X(fs) = f\nabla_X s + df(X)s$.

Example 12.1. *$\mathcal{E} = E \times M$ the trivial bundle, then $\Gamma(\mathcal{E}) = C^\infty(M, E)$. The trivial connection on $E \times M$ is $D_X\sigma := (D\sigma)(X)$. For any $f \in C^\infty(M)$, $D_{fX}\sigma = (D\sigma)(fX) = fD\sigma(X)$. $D_X(f\sigma) = (Df\sigma)(X) = df(X)\sigma + fD_X\sigma$.*

Let $\psi_1, \cdots, \psi_n$ be functions on M and $\sum \psi_1 = 1$. Let $\nabla^1, \cdots, \nabla^n$ be connection on M, then $\nabla := \sum \psi_i \nabla^i$ is also a connection: $\nabla_X\sigma = \sum \psi_i \nabla^i_X \sigma$.

Proposition 12.1. *If ∇ is a connection, $(\nabla_X\sigma)_m$ only depends on X, σ on a $V(m)$.*

Proof. Let $X_1 = X_2$ on a $V(m)$ and $\sigma_1 = \sigma_2$ on a $V(m)$. Let $\psi \equiv 1$ on $V(m)$ and $\operatorname{Supp}\psi \subset U$, then $\psi X_1 = \psi X_2$ and $\psi\sigma_1 = \psi\sigma_2$.

$$\nabla_{X_1}\sigma_1 = \nabla_{\psi X_1}\psi\sigma_1 = \nabla_{\psi X_2}\psi\sigma_2 = \nabla_{X_2}\sigma_2.$$

□

Proposition 12.2. *Every vector bundle admits a connection.*

Proof. Let $\{U_i\}_{i\in I}$ be a trivializing cover on M, i.e. $|_{\mathcal{U}_\rangle} \approx \mathcal{E} \times \mathcal{U}_\rangle$. Let $\{\psi_i\}_{i\in I}$ a partition of unity associated to U_i.

Finally let D^i be the trivial connection on $\mathcal{E}|_{U_i}$, then we define if $X \in \chi(M)$, $\sigma \in \Gamma(\mathcal{E})$,

$$\nabla_X\sigma = \sum_{i=1}^n \psi_i(D^i_X\sigma).$$

This is well-defined since $\operatorname{Supp}\psi_i \subset U_i$, and ∇ is a connection on $\mathcal{E}$. □

Definition 12.2. *Difference between two connection.*

Let ∇_1 and ∇_2 be two connections on $\mathcal{E} \to M$, then there exists $A \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ such that

$$\nabla^1_X \sigma - \nabla^2_X \sigma = A(X)\sigma.$$

Proof. Let define $B : TM \times \mathcal{E} \to \mathcal{E}$, $B(X, \sigma) = \nabla^1_X \sigma - \nabla^2_X \sigma$. Then $B(fX, \sigma) = fB(X, \sigma)$ and

$$B(X, f\sigma) = \nabla^1_X (f\sigma) - \nabla^2_X (f\sigma) = fB(X, \sigma) + df(X)\sigma - df(X)\sigma = fB(x, \sigma).$$

Then B is a tensor. By the lemma below, there exists a section A of the bundle

$$TM^* \otimes \mathcal{E}^* \otimes \mathcal{E} = TM^* \otimes \mathrm{End}(\mathcal{E}) = \Omega^1(M, \mathrm{End}(\mathcal{E})).$$

$\square$

Lemma 12.1. *Let $\mathcal{E}_1, \cdots, \mathcal{E}_k, \mathcal{F}$ be vector bundles over M. Let ψ be a k-multilinear map*

$$\psi : \Gamma(\mathcal{E}_1) \times \cdots \times \Gamma(\mathcal{E}_k) \to \Gamma(\mathcal{F}),$$

such that for any i, any $f \in C^\infty(M)$,

$$\psi(\sigma_1, \cdots, f\sigma_i, \cdots, \sigma_k) = f\psi(\sigma_1, \cdots, \sigma_k).$$

Then there exists C, a section of

$$\mathcal{E}_1^* \otimes \cdots \otimes \mathcal{E}_k^* \otimes \mathcal{F} = \mathcal{G},$$

such that

$$\psi(\sigma_1, \cdots, \sigma_k)_m = C_m((\sigma_1)_m, \cdots (\sigma_k)_m).$$

*We say ψ is a **tensor**.*

Proof. If $\sigma_i = \sigma_i'$ on $V(m)$, then $\psi(\sigma_1, \cdots, \sigma_k) = \psi(\sigma_1', \cdots, \sigma_k')$ on $V(m)$. (Repeat the proof about $\nabla_X \sigma$.)

It is enough to prove the result on $V(m)$, that is when $\mathcal{E}_i = E_i \times U$ and $\mathcal{F}_i = F \times U$. Let $(a^i_j)_{j \in I_i}$ be a basis of E_i, $\sigma_i = \sum\limits_j f^j_i a^i_j$, then

$$\psi(\sigma_1, \cdots, \sigma_k)_m = \sum \prod_{l=1}^{k} f^{j_l}_k (m) \psi(a^1_{j_1}, \cdots, a^k_{j_k}).$$

Then we define $C : M \to \mathcal{E}_1^* \otimes \cdots \otimes \mathcal{E}_k^* \otimes \mathcal{F}$ as

$$C_m = \sum a^{1,*}_{j_1} \otimes \cdots \otimes a^{k,*}_{j_k} \otimes \psi(a^1_{j_1}, \cdots, a^k_{j_k}).$$

$\square$

Proposition 12.3. *If ∇ is a connection and $A \in \Omega^1(M, \mathrm{End}(TM))$, then*

$$\nabla + A : (X, \sigma) \mapsto \nabla_X \sigma + A(X)\sigma$$

is also a connection. Then the space of connection is an affine space.

Proposition 12.4. *Let X be a vector field, σ a section of $\mathcal{E}$, let $m \in E$ and c a curve in M with $\dot{c}(0) = X_m$, then $(\nabla_X \sigma)_m$ only depends on X_m and the restriction of σ along c.*

Proof. Locally, $\nabla = D + A$, where D is the trivial connection on U and $A \in \Omega^1(U, \mathrm{End}(\mathcal{E}))$.

$$(\nabla_X \sigma)_m = (D_X \sigma)_m + (A(X)\sigma)_m,$$

the latter one only depends on X_m and σ_m since it is a tensor. And $(D_X\sigma)_m = (D_m\sigma)(X) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\sigma \circ c(t))$. □

Theorem 12.1 (Existence of Parallel Transport)**.** *Let $c(t) : [a, b] \to M$ be a curve on M, let $u \in \mathcal{E}_{c(0)}$, then there exists a unique $u(t)$ section of $\mathcal{E}$ along c, such that*

$$\nabla_{\dot{c}(t)} u(t) = 0.$$

Moreover, (i) if $k \in \mathbb{R}$, then $(ku)(tz) = k(u(t))$, (ii) if u and v are two vectors in $\mathcal{E}_{c(0)}$ then $(u + v)(t) = u(t) + v(t)$ (i.e. linear map from $\mathcal{E}_{c(0)}$ to $\Gamma(c^*\mathcal{E})$).

$u(t)$ is called the ***parallel transport*** *of u along $c(t)$.*

Proof. Let U be a neighborhood of $c(t_0)$ on which $\mathcal{E}$ is trivial $\mathcal{E} = E \times U$.

$$\nabla_{\dot{c}(t)} u(t)\big|_{t=t_0} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} (u \circ c(t)) + A(\dot{c}(t_0))u(t_0),$$

where $\nabla = D + A$. Check that this does not depend on the choice of trivialization.

To prove the existence and uniqueness of $u(t)$, it is enough to work locally solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) + A(\dot{c}(t))u(t) = 0,$$

with the initial condition $u(0) = u$. This is a consequence of the existence and uniqueness of solution of ODE on $[a, b]$. This is a linear equation hence (i) and (ii) holds.

Remark: The well-definition of A on c comes from the independent of the choice of trivialization. □

Definition 12.3. *We define the* ***Holonomy*** *linear map $\mathrm{Hol}_c : \mathcal{E}_{c(0)} \to \mathcal{E}_{c(1)}$, $u \mapsto u(1)$ where $u(t)$ is the parallel transport of u along $c(t)$.*

If α and β are two curves and $\alpha(1) = \beta(0)$, then

$$\mathrm{Hol}_\beta \circ \mathrm{Hol}_\alpha = \mathrm{Hol}_{\beta * \alpha}.$$

Theorem 12.2. Hol_c *is a linear isomorphism, for $c : [0, 1] \to M$*

Proof. Let $\tilde{c} : [0,1] \to M$, $\tilde{c}(t) = c(1-t)$. Let $u(t)$ be parallel section along c, then $\tilde{u}(t) = u(1-t)$ is a parallel section along $\tilde{c}$. In particular,

$$(\mathrm{Hol}_c)^{-1} = \mathrm{Hol}_{\tilde{c}}\,.$$

□

Recall the pull-back of vector bundle. Define $\psi : X \to Y$, let $\pi : \mathcal{E} \to Y$, we defined the vector bundle $\psi^*\mathcal{E} \to X$ as

$$\psi^*\mathcal{E} = \{(u,x) \in \mathcal{E} \times X : u \in \mathcal{E}_{\psi(x)}\}.$$

If $(U_i)_{i\in I}$ is a trivializing cover of $\mathcal{E}|_{U_i} = E \times U_i$ with cocycle $U_i \cap U_j \xrightarrow{\varphi^{ij}} GL(E)$. Then $(\psi^{-1}(U_i))_{i\in I}$ is a trivializing cover of X for $\psi^*\mathcal{E}$.

$$\psi^{-1}(U_i) \cap \psi^{-1}(U_j) \xrightarrow{\psi} U_i \cap U_j \xrightarrow{\varphi^{ij}} GL(E).$$

$\psi^*\varphi^{ij}$ are the transition functions. These satisfy the cocycle condition hence $\psi^*\mathcal{E}$ is a bundle.

What happens in a trivialization? In general if $\mathcal{O}_i$ is trivialization of $\mathcal{E} \to Z$, i.e. $\mathcal{E}|_{\mathcal{O}_i} \approx E \times \mathcal{O}_i$. A section $\sigma|_{\mathcal{O}_i}$ is a section of $\mathcal{E}|_{\mathcal{O}_i}$, $\mathcal{O}_i \to E$ with the compatibility condition on $\mathcal{O}_i \cap \mathcal{O}_j$, $\sigma_j(x) = \psi^{ji}(x)\sigma_i(x)$.

Induced section $\psi^* : \Gamma(\mathcal{E}) \to \Gamma(\psi^*\mathcal{E})$, $\sigma \mapsto \psi^*\sigma$, we can $\sigma_i^* = \sigma_i \circ \psi$. Then σ_i^* satisfies the compatibility condition with respect to $\psi^*\varphi^{ij}$.

Remark 12.1. *Not all sections of ψ^*E is induced sections. For example $X \to Y = \{0\}$.*

Definition 12.4. *Induced connection.*

Let $\mathcal{E} \to Y$ be a vector bundle over Y, ∇ be a connection on $\mathcal{E}$, and $\psi : X \to Y$. There exists a unique connection $\psi^*\nabla$ on $\psi^*\mathcal{E}$ such that if $u \in T_mX$, σ is a section of $\mathcal{E}$ defined on $V(\psi(m))$, then

$$[(\psi^*\nabla)_u(\psi^*\sigma)]_m = \psi^*[\nabla_{T\psi(u)}\sigma]_{\psi(m)}.$$

$(\psi^*\nabla)_u(\psi^*\sigma) = \psi^*(\nabla_{\psi_*u}\sigma)$.

Proof. Let ∇^1 and ∇^2 be two connection satisfying $\nabla^1_u(\psi^*\sigma) = \nabla^2_u(\psi^*\sigma) = \psi^*(\nabla_{\psi_*u}\sigma)$. Let write $\nabla^1 - \nabla^2 = A \in \Omega^1(X, \mathrm{End}(\psi^*\mathcal{E}))$. Thus A satisfies $A(u) = 0$ for every $u \in TX$, thus $A = 0$, hence $\nabla^1 = \nabla^2$.

Existence part: First case assume that $\mathcal{E} \to Y$ is trivial, $\mathcal{E} = E \times Y$. Any connection on E is $D + B$, where $B \in \Omega^1(Y, \mathrm{End}(E))$. Now $\psi^*\mathcal{E} = E \times X$. Let us define ∇^1 on $E \times X$, $\nabla^1 = D + \psi^*B$, where $\psi^*B(u) = B(T\psi(u))$.

Let us check that

$$\nabla^1_u(\psi^*\sigma) = D_x(\sigma \circ \psi)(u) + (\psi^*B)(u)(\sigma(\psi(x)) = (\nabla_{T\psi(u)}\sigma)_{\psi(x)}.$$

For the general case, take a trivializing cover of $Y = \{U_i\}$, then on $\psi^{-1}(U_i) \subset X$, we define $\nabla^i = \psi^*(\nabla|_{U_i})$. Then by uniqueness, $\nabla^i = \nabla^j$ on $\psi^*(\nabla|_{U_i}) \cap \psi^*(\nabla|_{U_j})$. We define $\psi^*\nabla = \nabla_i$ on U_i. □

Now a section along $c : [a,b] \to M$ is a section of the bundle $c^*\mathcal{E} \to [a,b]$, $u(t) = c^*u(t)$. We make it more clear,

$$\nabla_{\dot{c}(t)} u(t) = 0 \iff (c^*\nabla)_{\partial_t} c^* u = 0.$$

Theorem 12.3. *Let $\mathcal{E} \to [a,b]$ be a vector bundle with a connection ∇, then given any $u \in \mathcal{E}_a$, there exists a unique section $u(t)$ such that*

$$u(a) = u, \quad \nabla_{\partial_t} u(t) = 0.$$

Proof. It is enough to prove this result on $[t_0 - \varepsilon, t_0 + \varepsilon]$. Given $u \in \mathcal{E}_{t_0}$, there exists $u(t)$ such that $u(t_0) = u$ and $\nabla_{\partial_t} u = 0$ for any $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$.

Choose ε such that $\mathcal{E}|_{[t_0 - \varepsilon, t_0 + \varepsilon]}$ is trivial: $E \times [t_0 - \varepsilon, t_0 + \varepsilon]$. Set $\nabla = D + B$ and

$$\nabla_{\partial_t} u = 0 \iff \frac{\mathrm{d}}{\mathrm{d}t} u(t) + B_t(\partial_t) u(t) = 0.$$

In other words, the curve $u(t) \in E$ satisfies

$$\dot{u}(t) + C(t) \cdot u(t) = 0,$$

where $C(t) = B_t(\partial_t) : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathrm{End}(E)$. □

Now we have finished the proof of the existence of parallel transport.

Corollary 12.1. *Any bundle $\mathcal{E} \to \mathbb{R}$ is trivial.*

Proof. Let $u_1, \cdots, u_k$ be a basis of $\mathcal{E}_0$, define $\mathbb{R}^k \times \mathbb{R} \to \mathcal{E}$,

$$\big((a_1, \cdots, a_k), t\big) \mapsto \sum a_i u_i(t),$$

where $u_i(t)$ is the parallel transport of u_i. □

Exercise 12.1. *Every vector bundle over a contractible open set is trivial.*

12.2 Connection and curvature

Another point of view on parallel transport: Horizontal distribution.

Let $\mathcal{E} \to M$ and $\mathcal{E}$ itself is a manifold.

Definition 12.5. *An **horizontal distribution** is a distribution $\mathcal{F}_u \subset T_u\mathcal{E}$ such that*

$$T\pi : \mathcal{F}_u \to T_{\pi(u)} M,$$

is an isomorphism.

$\mathrm{Ker}(T_u\pi) = T_u(\mathcal{E}_{\pi(u)})$ *is called the **vertical distribution** V_u.*

We have (i) $\dim \mathcal{F}_u = \dim M$, $\mathcal{F}_u \cap \mathrm{Ker}(T\pi) = \{0\}$.

$\mathcal{F}_u$ is an trivialization distribution iff $\forall u \in \mathcal{E}$, $\mathcal{F}_u \oplus V_u = T_u\mathcal{E}$.

Proposition 12.5. *A connection ∇ on $\mathcal{E}$ defines an horizontal distribution $\mathcal{F}^\nabla$ such that, for $u : [a,b] \to \mathcal{E}$, $\dot{u}(t) \in \mathcal{F}^\nabla_{u(t)}$ iff $u(t)$ is parallel along the curve $c(t) = \pi u(t)$.*

Example 12.2. *$\mathcal{E} = E \times M$ and D is the trivial connection. $u(t)$ is parallel along $c(t)$,*
iff $u(t)$ is constant along $c(t)$, as a map $[a,b] \to E$,
iff curve $u(t)$ is tangent to the distribution $\{0\} \times TM$ of $T(E \times M)$,
iff u is tangent to the distribution $\mathcal{F}_{(v,m)} = \{m\} \times T_m M$.

Proof of proposition. Consider a trivialization $\mathcal{E}|_{U_i} = E \times U_i$ and $\nabla = D + A$. Let $u : [a,b] \to \mathcal{E}|_{U_i}$, $u(t) = \big(v(t), c(t)\big)$, where $v(t) \in E$ and $c(t) \in U_i$.

$$\nabla_{\dot{c}} u(t) = \big(\frac{\mathrm{d}}{\mathrm{d}t} v(t) + A_{c(t)}(\dot{c}(t)) \cdot v(t), c(t)\big),$$

hence $\nabla_{\dot{c}} u(t) = 0$ iff $\dot{v}(t) + A_{c(t)}(\dot{c}(t)) \cdot v(t) = 0$.

Let

$$\mathcal{F}^\nabla_{(v,x)} = \{(w, \gamma) \in E \times T_x M : w + A_x(\gamma) v = 0\}.$$

$\big(v(t), c(t)\big) \in \mathcal{E}|_U = E \times U$ is tangent to $\mathcal{F}^\nabla$, iff $\big(\dot{v}(t), \dot{c}(t)\big) \in \mathcal{F}^\nabla$, iff $\dot{v}(t) + A_{c(t)}(\dot{c}(t)) \cdot v(t) = 0$, iff $\nabla_{\dot{c}} u(t) = 0$. □

$\mathcal{F}^\nabla$ is integrable, iff ∇ is locally trivial, iff there exists trivialization $\mathcal{E} = E \times U$ in which $\nabla = D$, iff $R^\nabla = 0$, $R^\nabla \in \Omega^2(M, \mathrm{End}(E))$.

In general, $\omega = \mathrm{Tr}\big(R^\nabla \wedge \cdots \wedge R^\nabla\big)$ is a closed $2n$ form on M. Its cohomology class only depends on $\mathcal{E} \to M$.

Lemma 12.2. *∇ on $\pi : \mathcal{E} \to M$ and $X \in \chi(M)$. There exist a unique vector field Y on $\mathcal{E}$ such that (i) Y is horizontal ($Y_u \in \mathcal{F}^\nabla_u$), (ii) $(T_u \pi)(Y_u) = X_{\pi(u)}$.*

Definition 12.6. *Y is the **horizontal lift** of X.*

Proof. $T_u \pi$ is an isomorphism between $\mathcal{F}^\nabla_u$ and $T_x M$. Locally, $\mathcal{E} = E \times M$ trivial, then $Y_u = (-A_x(X)u, X)$. □

Lemma 12.3. *Let $X \in \chi(M)$ with flow φ_t, $Y \in \chi(\mathcal{E})$ with flow ψ_t is the horizontal lift of X. Then (i) $\pi \circ \psi_t = \varphi_t \circ \pi$, (ii) $t \mapsto u(t) = \psi_t(u)$ is parallel along the curve $c(t) = \varphi_t(x)$.*

Proof. Let $u \in \mathcal{E}$ and $x = \pi(u)$. Define $\tilde{c}(t) = \pi \circ \psi_t(u)$, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} \tilde{c}(s) = (T_{\psi_t(u)}\pi) \left(\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} \psi_s(u) \right) = T_{\psi_t(u)}\pi(Y_{\psi_t(u)}) = X_{\tilde{c}(t)}.$$

Then $\tilde{c}(t)$ is an orbit of X, hence $\tilde{c}(t) = \varphi_t(x)$. □

Definition 12.7. *A connection ∇ is **flat** if $\nabla = D$ in a local trivialization.*

Theorem 12.4. *$\mathcal{F}^\nabla$ is integrable iff ∇ is flat.*

Proof. If ∇ is flat. In a trivialization $\nabla = D$ and $\mathcal{E} = E \times M$. The horizontal distribution is

$$\mathcal{F}^{\nabla}_{(u,m)} = \{(-A(x)u, X) : X \in T_m M\} = \{0\} \times T_m M.$$

In that case the horizontal distribution is integrable.

Assume $\mathcal{F}^{\nabla}$ is integrable. Let $\mathscr{O} \subset M$ and $\mathscr{O} \approx]-1,1[^n \subset M$. Let $X_i = \frac{\partial}{\partial x_i}$ the coordinate vector fields. Let Y_i be the horizontal lifts of X_i, let φ^i_t the flow of X_i and ψ^i_t the flow of Y_i.

Lemma 12.4. $[Y_i, Y_j] = 0$ *(iff $\psi^i_t \circ \psi^j_s = \psi^j_s \circ \psi^i_t$).*

Let $u \in \mathcal{E}$, let $N \ni u$ be the submanifold such that $\forall n \in N$, $T_n N = \mathcal{F}^{\nabla}_n$. Hence $\forall n \in N$, $T\pi$ is an isomorphism from $T_n N$ to $T_{\pi(n)} M$. We can find an open set $\mathscr{O} \ni n$ in N such that $\pi : \mathscr{O} \to \pi(\mathscr{O})$ is a diffeomorphism (local immersion theorem). From now on, redefine $N = \mathcal{O}$. Set $\tilde{Y}_i = Y_i|_N$,

$$\pi_*[\tilde{Y}_i, \tilde{Y}_j] = [\pi_* \tilde{Y}_i, \pi_* \tilde{Y}_j] = [X_i, X_j] = 0.$$

This means that $T\pi([\tilde{Y}_i, \tilde{Y}_j]) = 0$, hence that $[\tilde{Y}_i, \tilde{Y}_j] = 0$ (since $\pi|_N$ is a diffeomorphism). $[Y_i, Y_j]_u = [\tilde{Y}_i, \tilde{Y}_j]_u$ because N is a submanifold and Y_i tangent to N.

Lemma 12.5. *If W a submanifold of M. If X, Y are vector fields on M, such that $X_w, Y_w \in T_w W$ for $w \in W$, then $[X, Y]|_W = [X|_W, Y|_W]$ (Hint: use a chart).*

Proof. Since W is a submanifold of M, for $w \in W \subset M$, there is a chart (U, φ) such that $\varphi(U \cap M) \subset \mathbb{R}^k \subset \mathbb{R}^n$. Set $X = X_i \frac{\partial}{\partial x_i}$ and $Y = Y_j \frac{\partial}{\partial x_j}$. Since $X(w)$ and $Y(w)$ is in $T_w W$ for $w \in M$, we know that $X_{k+1}(w) = \cdots = X_n(w) = 0$ and $Y_{k+1}(w) = \cdots = Y_n(w) = 0$ for any $w \in W$.

$$[X, Y] = \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} = (X(Y_j) - Y(X_j)) \frac{\partial}{\partial x_j}.$$

For $j \geq k+1$, since $Y_j|_W = 0$ and $X(w) \in T_w W$ for any $w \in W$, we have $X(Y_j)(w) = 0$. Ditto for $Y(X_j)(w) = 0$, $j \geq k+1$. Hence we've proved that

$$[X, Y]_w = \sum_{1 \leq j \leq k} (X(Y_j) - Y(X_j)) \frac{\partial}{\partial x_j} \in T_w W,$$

$\forall w \in W$. Note that $X|_W = \sum_{1 \leq j \leq k} X_i \frac{\partial}{\partial x_i}$ and $Y|_W = \sum_{1 \leq j \leq k} Y_i \frac{\partial}{\partial x_i}$ and $[X|_W, Y|_W] = [X, Y]|_W$. $\square$

Back to original proof. Set $\mathscr{O} = \times]-1,1[^n$, we define the trivialization $\mathcal{E}_0 \times \mathscr{O} \xrightarrow{\Phi} \mathcal{E}|\mathscr{O}$ by $\Phi(u, t_1, \cdots, t_n) = (\psi^1_{t_1} \circ \cdots \circ \psi_{t_n})(u)$.

We will prove that (i) Φ is linear, in particular, Φ^{-1} is a trivialization. (ii) Given a curve $c(s)$ in $\mathscr{O}$, $u \in \mathcal{E}_0$ then $\Phi(u, c(s))$ is horizontal.

We first prove (ii).

$$c(s) = \left(\varphi^1_{t_1(s)} \circ \cdots \circ \varphi^n_{t_n(s)}\right)(0).$$

$$u(s) := \Phi(u, c(s)) = \big(\psi^1_{t_1(s)} \circ \cdots \circ \psi^n_{t_n(s)}\big)(u).$$

From Lemma 12.3, $\pi u(s) = c(s)$. $\frac{\mathrm{d}}{\mathrm{d}u}\big|_{t=s} u(t) = \sum \dot{t}_i(s) Y_i(u(s))$. Indeed, we can put $\psi^i_{t_i}$ in the first place for $\frac{\partial}{\partial t_i}$ (See Proposition 10.6).

Now we have proved that $s \mapsto \Phi(u, c(s))$ is horizontal. $u \mapsto \Phi(u, c(s_0)) = Hol_{c(s_0)}(u)$ is linear. Now $\mathcal{F}^\nabla_u = \{(0, X) : X \in TM\}$ hence ∇ is flat. □

Definition 12.8. *Given ∇ a connection on $\pi : \mathcal{E} \to M$, the curvature tensor $\chi(M) \times \chi(M) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ by*

$$R^\nabla(X, Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma.$$

Lemma 12.6. *(i) $R(X, Y)\sigma = -R(Y, X)\sigma$.*

(ii) $R(fX, Y)\sigma = fR(X, Y)\sigma$, for $f \in C^\infty M$.

(iii) $R(X, Y)f\sigma = fR(X, Y)\sigma$, for $f \in C^\infty M$.

Corollary 12.2. *Given ∇, there exists $R^\nabla_0 \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$ such that*

$$\big(R^\nabla(X, Y)\sigma\big)_m = (R^\nabla_0)_m(X_m, Y_m)\sigma_m.$$

In a trivialization, $\mathcal{E}|_U = E \times U$ and $\nabla = D + A$.

Lemma 12.7. $R^\nabla(X, Y)\sigma = dA(X, Y)\sigma + [A(X), A(Y)]\sigma$.

Proof.

$$R^\nabla(X, Y)\sigma = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma.$$

$$\nabla_{[X,Y]}\sigma = L_{[X,Y]}\sigma + A([X, Y])\sigma.$$

$$\begin{aligned}
\nabla_X \nabla_Y \sigma &= \nabla_X(D\sigma(Y) + A(Y)\sigma) \\
&= \nabla_X(L_Y\sigma) + (L_X A(Y))\sigma + A(Y)L_X\sigma \\
&= L_X L_Y \sigma + A(X)L_Y\sigma + +(L_X A(Y))\sigma + A(Y)L_X\sigma + A(X)A(Y)\sigma
\end{aligned}$$

Similarly,

$$\nabla_Y \nabla_X \sigma = L_Y L_X \sigma + A(Y)L_X\sigma + +(L_Y A(X))\sigma + A(X)L_Y\sigma + A(Y)A(X)\sigma.$$

With the fact, if $\omega \in \Omega^1(M)$, then $d\omega(X, Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X, Y])$, we've done the proof. □

Theorem 12.5. *∇ is flat, iff $R^\nabla = 0$.*

Proof. If ∇ is flat, then $A = 0$ hence $R^\nabla = 0$.

If $R^\nabla = 0$, show that $\mathcal{F}^\nabla$ is integrable. Let $X_i = \frac{\partial}{\partial x_i}$ be coordinate vector fields on V and Y_i be the horizontal lift.

$$(Y_i)_u = \big(-A(X_i) \cdot u, X_i\big).$$

Lemma 12.8. *If $R^\nabla = 0$, then $[Y_i, Y_j] = 0$.*

Consequence $\mathcal{F}^\nabla$ is integrable and thus ∇ flat.

$$\begin{aligned}
[Y_i, Y_j]_u &= [\big(-A(X_i)\cdot u, X_i\big), \big(-A(X_j)\cdot u, X_j\big)] \\
&= (-\frac{\partial}{\partial x_i}A(X_j)u + \frac{\partial}{\partial x_j}A(X_i)u - [A(X_i), A(X_j)u], 0) \\
&= (-dA(X_i, X_j)u - [A(X_i), A(X_j)]u, 0) \\
&= (-R^\nabla(X_i, X_j)u, 0).
\end{aligned}$$

Here we use the fact $[Au, Bu] = -[A, B]u$ (See Exercise 10.4).

□

Theorem 12.6. *If ∇ is flat, if $c(t)$ is homotopic to $\tilde{c}(t)$ with fixed endpoints, then*

$$Hol_c = Hol_{\tilde{c}} : \mathcal{E}_{c(0)} \to \mathcal{E}_{c(1)}.$$

Let $\mathcal{E}$ be a $\mathbb{R}$-vector bundle with a connection ∇.

$$\hat{P}_k^\nabla(X_1, \cdots, X_{4k}) = \sum_{\sigma\in S_{2k}} (-1)^\sigma \operatorname{Tr}\big(R^\nabla(X_{\sigma(1)}, X_{\sigma(2)})\cdots R^\nabla(X_{\sigma(4k-1)}, X_{\sigma(4k)})\big)$$

Let $\mathcal{E}$ be a $\mathbb{C}$-vector bundle, $J \in \gamma(\mathcal{E})$ such that $J^2 = -1$.

$$\hat{c}_k^\nabla(X_1, \cdots, X_{2k}) = \sum_{\sigma\in S_k} (-1)^\sigma \operatorname{Tr}\big(R^\nabla(X_{\sigma(1)}, X_{\sigma(2)})\cdots R^\nabla(X_{\sigma(2k-1)}, X_{\sigma(2k)})\big)$$

Theorem 12.7. *The forms $\hat{P}_k^\nabla$ and $\hat{c}_k^\nabla$ are closed (called Pontryagin, Chern classes). Their cohomology classes $[\hat{P}_k^\nabla] \in H^{4k}(M)$ and $[\hat{c}_k^\nabla] \in H^{2k}(M)$ only depends on $\mathcal{E}$ are called the Pontryagin class and the Chern class.*

Example 12.3. *$T\mathbb{S}^2$ has complex structure and $c_1(T\mathbb{S}^2) \neq 0$. Hence $T\mathbb{S}^2$ is not trivial.*

13 Group action

13.1 Properly discontinuous action without fixed points

A covering is a map $p : X \to Y$, where X and Y are topological spaces, such that for any $y \in Y$, there exists $U \in V(y)$ such that $p^{-1}(U) = \bigsqcup_{z \in Z} \mathcal{O}_z$, where $\mathcal{O}_z$ are open sets and $p : \mathcal{O}_z \to U$ is a homeomorphism.

If Γ is a group acting on X, with a **properly discontinuous action without fixed points** that

(i) $\forall \gamma \in \Gamma$, $\gamma : x \to \gamma x$ is a diffeomorphism (homeomorphism).

(ii) $\forall x \in X$, there exists $U \in V(x)$, such that $\gamma U \cap U \neq \varnothing$ will indicate $\gamma = \mathrm{id}$.

Remark 13.1. ***properly discontinuous action.***

For every compact set K, $\#\{\gamma : \gamma K \cap K \neq \varnothing\} < \infty$.

Theorem 13.1. *In topology category, there is a topology on $\Gamma \backslash X$, such that $p : X \to \Gamma \backslash X$ is a covering.*

Proof. Recall the quotient topology on $\Gamma \backslash X$ is give by U is an open set in $\Gamma \backslash X$ if $p^{-1}(U)$ is open.

We define $U_\gamma := \gamma \cdot U$ which is an open set and $U_\gamma \cap U_{\gamma'} = \varnothing$ if $\gamma \neq \gamma'$.

We want to show p is covering: for any $y \in \Gamma \backslash X$, there exists $V \in V(y)$ such that $p^{-1}(V) = \bigsqcup_{z \in Z} V_z$ and $p : V_z \to V$ is a homeomorphism.

Let $y \in \Gamma \backslash X$, let x such that $p(x) = y$. Let $U \in V(x)$ such that $\gamma \neq \gamma'$ in Γ, $U_\gamma \cap U_{\gamma'} = \varnothing$. Let $V = p(U)$, then

$$p^{-1}(V) = p^{-1}(p(U)) = \bigsqcup_{\gamma \in \Gamma} U_\gamma,$$

is an open set hence V is open in $\Gamma \backslash X$. Indeed, let $z \in p^{-1}(p(U))$, $p(z) \in V = p(U)$, then $p(z)$ is an orbit $\{\gamma \alpha\}_{\alpha \in \Gamma}$ which intersects U with $\alpha \in U$.

Then we also need to check that $p : U_\gamma \to V$ is a homeomorphism. It is obvious that this map is bijective and continuous. We only need to show that $p|_{U_\gamma}^{-1}$ is continuous, which is similar to the proof of the openness of V. □

Remark 13.2. *We say U is good if U is an open set in X such that $\forall \gamma \neq \mathrm{id}$, $\gamma U \cap U = \varnothing$.*

Theorem 13.2. *In differential geometry category, there is a manifold structure on $\Gamma \backslash X$, such that $p : X \to \Gamma \backslash X$ is a covering, p is smooth and a local diffeomorphism.*

Proof. X is σ-compact then $\Gamma \backslash X$ is σ-compact.

We can find an atlas $\{(U_i, \phi_i)\}$ of X such that U_i are good. Because U_i is good: $U_i \to p(U_i)$ is a homeomorphism. We define charts of $\Gamma \backslash X$ by $\{(V_i = p(U_i), \phi_i \circ p_i^{-1} = p|_{U_i}^{-1}\}$. It remains to prove these charts are compatible.

For any $y \in \Gamma\backslash X$, and $y \in V_i \cap V_j$, we say $x_i = p_i^{-1}(y)$ and $x_j = p_j^{-1}(y)$. notice that x_i may not equal to x_j, but we have an element $\gamma \in \Gamma$ such that $x_j = \gamma x_i$. The transition map is

$$(\phi_j \circ p_j^{-1}) \circ \left(\phi_i \circ p_i^{-1}\right)^{-1} = \phi_j \circ p_j^{-1} \circ p_i \circ \phi_i^{-1} = \phi_j \circ \gamma \circ \phi_i^{-1},$$

around y, hence differential. □

Exercise 13.1. *Check* $\Gamma\backslash X$ *is Hausdorff.*

Exercise 13.2. Γ *is a finite group and* X *is compact, show that if for any* $\gamma \in \Gamma\backslash\,\mathrm{id}$, *and for any* x, *we have* $\gamma x \neq x$, *then* Γ *acts properly discontinuously on* X.

Example 13.1. $\Gamma = \mathbb{Z}/2\mathbb{Z} = \{-1, 1\}$. Γ *acts on* $S^n = \{u \in \mathbb{R} : \|u\| = 1\}$. $\Gamma\backslash S^n$ *is* $\mathbb{P}^n$.

13.2 Action of group of diffeomorphisms

Theorem 13.3. *Let* M *be a connected manifold. Let* $\mathscr{G} = \{$*diffeomorphism of* $M\}$. *For all* $p \in \mathbb{N}$, $\mathscr{G}$ *acts transitively on* $M^{p*} := \{(m_1, \cdots, m_p) : \forall i \neq j, m_i \neq m_j\}$.

That is, given $(m_1, \cdots, m_p), (q_1, \cdots, q_p) \in M^{p*}$, then there is a diffeomorphism φ of M such that $\varphi(m_i) = q_i$ for any $1 \leq i \leq p$.

1. Show that if B is the open ball in $\mathbb{R}^n$. Given $x \in B$, there is a $\mathscr{O} \in V(x)$, for any $y \in \mathscr{O}$, there is φ a diffeomorphism of B such that (i) $\varphi(x) = y$ and (ii) φ is an identity on a neighborhood on ∂B^n.

   For $\varphi(x) = y$ we first define the transition map $\varphi_0(z) = z + u$, where $u = y - x$ small enough.

   Let ψ be the function such that ψ is 1 on a neighborhood of x and 0 on a neighborhood of ∂B^n. We define $\varphi(z) = z + \psi(z)u$.

   Since $D_z\varphi = I + uA_{\psi,z}$, for u small enough, $D_z\varphi$ is non-singular. Hence φ is local diffeomorphism.

   Now we prove that for u small enough, φ is injective. If $\varphi(z_1) = \varphi(z_2)$, i.e. $z_1 + \psi(z_1)u = z_2 + \psi(z_2)u$. We know that there exists K_0 such that ψ is K_0-Lipschitz, that is
   $$|\psi(z_1) - \psi(z_2)| \leq K_0\|z_1 - z_2\|.$$
   Then we just need to choose $\|u\| \leq \frac{1}{2K_0}$.

   Since φ is local diffeomorphism and injective, we say it is a diffeomorphism.

   We can only say that $\varphi : B \to \varphi(B)$ is a diffeomorphism. We also need to prove that φ is surjective. We can prove it by flow of diffeomorphism. xxx.

2. Let us fix $m = (m_1, \cdots, m_p) \in M^{p*}$,
   $$\mathscr{G}_m = \{(q_1, \cdots, q_p) \in M^{p*} : \exists\varphi \text{ diffeomorphism such that } \varphi(m_i) = q_i\}.$$

Actually $\mathscr{G}_m$ is just an orbit of $\mathscr{G}$. We will prove $\mathscr{G}_m$ is open.

For any i, let $\mathscr{O}_i$ be a neighborhood of q_i such that

(i) $\mathscr{O}_i \cap \mathscr{O}_j$ if $i \neq j$.

(ii) $\mathscr{O}_i$ is diffeomorphism to a ball.

(iii) Let $U_i \subset \mathscr{O}_i$ with the following property. For any $z_i \in U_i$, there is a diffeomorphism φ_i such that $\varphi_i(q_i) = z_i$ and φ_i is the identity on neighborhood $\mathscr{V}_i$ of $\partial \mathscr{O}_i$.

Let $\varphi : M \to M$ defined by $\varphi = \varphi_i$ on $\mathscr{O}_i$ and $\varphi = \mathrm{id}$ on $M \setminus \sqcup(\mathscr{O}_i \setminus \mathscr{V}_i)$. Then φ is a diffeomorphism.

Now we've proved that for any $(q_1, \cdots, q_p) \in \mathscr{O}_m$, there is a neighborhood $U_1 \times \cdots \times U_p \subset M^{p*}$, for any $z_i \in U_i$ there is a diffeomorphism ψ such that $\psi(q_i) = z_i$. Since $(q_1, \cdots, q_p) \in \mathscr{G}_m$, there is φ such that $\varphi(m_i) = q_i$, hence $\psi \circ \varphi(m_i) = z_i$, i.e. $(z_1, \cdots, z_p) \in \mathscr{G}_m$.

3. $\mathscr{G}_m$ is closed. Note that what we've proved is that every orbit of $\mathscr{G}$ on M^{p*} is open. Then
$$\mathscr{G}_m = M^{p*} \setminus \bigcup_{q \notin \mathscr{G}_m} \mathscr{G}_q.$$

4. We will show that for $\dim M \geq 2$, M^{p*} is connected. Hence $\mathscr{G}_m = M^{p*}$.

Example 13.2. *Counter example for M not connected.*

Set $M = \mathbb{S}^2 \sqcup \mathbb{R}^2$, the diffeomorphism group is not transitive. Since a diffeomorphism group sends a connected component to a connected component and a compact set to compact set. Then a point in $\mathbb{S}^2$ can only be mapped into $\mathbb{S}^2$.